Problem Solving, Fuzzy Logic and Computational Thinking

Michael Gr. Voskoglou

School of Technological Applications
Graduate Technological Educational Institute of Patras, GREECE
mvosk@hol.gr , http://eclass.teipat.gr/eclass/courses/523102

Abstract

Computational thinking (CT) is a new problem solving method named for its extensive use of computer science techniques. However, it is the modelling thinking that constitutes the essence of CT, since it synthesises all the other components of CT for the solution of the corresponding problem.

In the present paper we develop a fuzzy model for the CT process by representing the main stages of the modelling process as fuzzy subsets of a set of linguistic labels characterizing the modellers' performance in each of these stages. We also apply the 'centroid' method in obtaining a measure of the individuals' CT skills. Two classroom experiments are presented illustrating the use of our fuzzy model in practice. The results of these experiments give a strong indication that the use of computers as a tool for problem solving enhances the students' abilities in solving real world problems involving mathematical modelling. This finding has been crossed by us and by other researchers in earlier papers.

Keywords: *Fuzzy sets, system's effectiveness, 'centroid' method, computational thinking, problem solving, mathematical modelling.*

1. Introduction

Computational thinking (CT) is a new problem solving method named for its extensive use of computer science techniques. It synthesizes critical thinking and existing knowledge and applies them to solve complex real world technological problems. The relationship between CT and critical thinking, the two modes of thinking in solving problems, has not been yet clearly established. In a recent paper [1] we attempted to shed some light into this relationship. In the same paper we also presented two classroom experiments the results of which suggest that the use of computers as a tool for problem solving enhances the students' abilities in solving real world problems involving mathematical modelling.

Actually, living in a knowledge era and an ever increasing progress in technology, combining knowledge and technology to solve problems is becoming the mode rather than the exception. However, the solution of a complex real world technological problem requires the construction and use of a *model* translating the objects or phenomena from the real world system into mathematical or computer relations (mathematical or simulation model respectively). Luckily, although a real situation may involve a substantial number of variables, generally only a small fraction of them truly dominates the behaviour of the system, thus affecting the solution of the corresponding problem. Therefore the simplification of the real system in terms of the model concentrates primarily on identifying the dominant variables and the relationships affecting them. Thus the construction of the model involves a deep abstracting process.

One could actually claim that *modelling thinking constitutes the essence of CT*, since it synthesises all the other components of CT (abstract, logical and constructive thinking; see [2] or section 2 of [1]) for the solution of the corresponding problem. In fact, it is well known (e.g. [3]; paragraph 1.4) that the main stages of the modelling process involve:

- *Analysis* of the given problem, i.e. understanding of its statement and recognizing limitations, restrictions and requirements of the real system (critical thinking).
- *Construction* of the model (abstract thinking).
- *Solution* of the model, achieved by proper logical manipulation (logical thinking).
- *Validation* (control) of the model, usually achieved by reproducing through it the behaviour of the real system under the conditions existing before the solution of the model (empirical results, special cases etc).
- *Implementation* of the final results to the real system, i.e. 'translation' of the solution obtained in terms of the model to the 'language' of the real situation in order to reach the required practical conclusions needed for the solution of the given real problem (constructive thinking).

The most important type of model in use is the *symbolic* or *mathematical model*. In formulating this type one assumes that all relevant variables are quantifiable. These variables are then related by the appropriate mathematical relations (functions, equations, inequalities, etc) to describe the behaviour of the system and the solution of the model is achieved by proper mathematical manipulation. In this case the stage of the construction of the model is usually called *mathematization* and presupposes the formulation of the real situation in so that it is ready for mathematical treatment (for more details see [4] and its references).

In addition to mathematical models, *simulation* and *heuristic* models are also used in cases where the mathematical formulation of the real situation is too complex to allow an exact solution. The former 'mimic', usually with the help of a properly designed computer program, the behaviour of the system over a period of time by specifying a number of events (as points in time) whose occurrence signifies that important information pertaining to the behaviour of the system can be gathered. Once such events are defined, it is necessary to pay attention to the system only when an event occurs. The main drawback of simulation is that the analysis is equivalent on conducting experiments and is thus subject to experimental error.

Heuristics are actually 'search' procedures relying on intuitive or empirical rules that, given a current solution to the model, allow the determination of an improved solution. When no further improvement can be achieved, the best attained solution is an approximate solution to the model (e.g. techniques of Numerical Analysis).

Notice that the stages of the modelling process presented above are helpful in understanding the modellers' *'ideal behaviour'*, in which they proceed from real world problems through a model to acceptable solutions and report on them. However, things in real situations are usually not happening like that. For example, recent research, ([5], [6], etc), reports that students in school take *individual modelling routes* when tackling mathematical modelling problems, associated with their individual learning styles. The human cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teacher's point of view there usually exists vagueness about the degree of students' success in each of the stages of the modelling process. All these gave us the impulsion to introduce principles of fuzzy sets theory in order to describe in a more effective way the process of modelling in particular and of CT in general. For general facts on fuzzy sets we refer freely to the book [7].

2. The Fuzzy Model

For the development of our fuzzy model for the modelling process we consider a group of n modellers, $n \geq 2$, working (each one individually) on the same modelling problem. In order to make our model technically simpler, we can, without loss of the generality, reduce the stages of the modelling process to the following three:

- S_1 : Analysis/Construction of the model.
- S_2 : Solution of the model.
- S_3 : Validation of the model/Implementation to the real system.

In fact, the analysis of the given problem is an introductory stage of the MM process that can be naturally seen as being a sub step of the construction of the model. Further, the stage of implementation of the final results to the real system is an expected action following the validation of the model, which means that the joined stage of Validation/Implementation can be actually considered as the final stage of the modelling process.

Denote by *a, b, c, d,* and *e* the linguistic labels of very low, low, intermediate, high and very high success respectively of a system's entity in each of the S_i*'s*. Set

$$U = \{a, b, c, d, e\}$$

We are going to attach to each stage S_i of the modelling process, i=1,2,3 , a fuzzy subset, A_i of U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of modellers that faced very low, low, intermediate, high and very high success at stage S_i respectively, i=*1,2,3,* we define the *membership function* m_{Ai} for each x in U, as follows:

$$1, \quad \text{if} \quad \frac{4n}{5} < n_{ix} \leq n$$

$$0.75, \quad \text{if} \quad \frac{3n}{5} < n_{ix} \leq \frac{4n}{5}$$

$$m_{A_i}(x) = \begin{cases} 0.5, & \text{if } \frac{2n}{5} < n_{ix} \le \frac{3n}{5} \\ 0.25, & \text{if } \frac{n}{5} < n_{ix} \le \frac{2n}{5} \\ 0, & \text{if } 0 \le n_{ix} \le \frac{n}{5} \end{cases}$$

In fact, if one wanted to apply 'probabilistic' standards in measuring the degree of success of the modellers at each stage of the process, then he/she should use the relative frequencies $\frac{n_{ix}}{n}$. Nevertheless, such an action would be highly questionable, since the n_{ix}'s are obtained with respect to the linguist labels of U, which are fuzzy expressions by themselves. Therefore the application of a fuzzy approach by using membership degrees instead of probabilities seems to be more suitable for this case. But, as it is well known, the membership function needed for such purposes is usually defined empirically in terms of logical or/and statistical data. In our case the above definition of m_{A_i} seems to be compatible with the common logic.

Then the fuzzy subset A_i of U corresponding to S_i has the form:

$$A_i = \{(x, m_{Ai}(x)): x \in U\}, i=1, 2, 3.$$

In order to represent all possible *profiles (overall states)* of the system's entities during the corresponding process we consider a *fuzzy relation*, say *R*, in U^3 of the form:

$$R= \{(s, m_R(s)): s=(x, y, z) \in U^3\}.$$

For determining properly the membership function m_R we give the following definition:

A profile s=(x, y, z), with x, y, z in U, is said to be well ordered if x corresponds to a degree of success equal or greater than y and y corresponds to a degree of success equal or greater than z.

For example, *(c, c, a)* is a well ordered profile, while *(b, a, c)* is not.
We define now the *membership degree* of a profile *s* to be

$$m_R(s) = m_{A_1}(x) m_{A_2}(y) m_{A_3}(z)$$

if *s* is well ordered, and *0* otherwise.

In fact, if for example the profile *(b, a, c)* possessed a nonzero membership degree, how it could be possible for a modeller, who has failed in solving the model, to perform satisfactorily at the validation of it?

Next, for reasons of brevity, we shall write m_s instead of $m_R(s)$. Then the *probability* p_s of the profile s is defined in a way analogous to crisp data, i.e. by

$$P_s = \frac{m_s}{\sum_{s \in U^3} m_s}$$

We define also the *possibility* r_s of *s* to be

$$r_s = \frac{m_s}{\max\{m_s\}} \quad ,$$

where $max\{m_s\}$ denotes the maximal value of m_s , for all *s* in U^3. In other words the possibility of *s* expresses the "relative membership degree" of *s* with respect to $max\{m_s\}$.

Assume further that one wants to study the *combined results* of behaviour of *k* different groups of a system's entities, $k \geq 2$, during the same process. For this, we introduce the *fuzzy variables* $A_1(t)$, $A_2(t)$ and $A_3(t)$ with $t=1, 2, ..., k$. The values of these variables represent fuzzy subsets of *U* corresponding to the stages of the modelling process for each of the *k* groups; e.g. $A_1(2)$ represents the fuzzy subset of *U* corresponding to the stage of Analysis/construction of the model for the second group ($t=2$). It becomes evident that, in order to measure the degree of evidence of the combined results of the *k* groups, it is necessary to define the probability *p(s)* and the possibility *r(s)* of each profile *s* with respect to the membership degrees of *s* for all groups. Therefore we introduce the *pseudo-frequencies*

$$f(s) = \sum_{t=1}^{k} m_s(t)$$

and we define the probability and possibility of a profile *s* by

$$p(s) = \frac{f(s)}{\sum_{s \in U^3} f(s)} \quad \text{and} \quad r(s) = \frac{f(s)}{\max\{f(s)\}} \quad \text{respectively,}$$

where $max\{f(s)\}$ denotes the maximal pseudo-frequency.

Obviously the same method could be applied when one wants to study the combined results of behaviour of a group during *k* different modelling situations.

The above model gives, through the calculation of probabilities and possibilities of all modellers' possible profiles, a quantitative view of their realistic performance in all stages of the modelling process.

3. Measuring model building and CT capacities

There are *natural* and *human-designed* real systems. In contrast to the former, which may not have an apparent objective, the latter are made with purposes that are achieved by the delivery of outputs. Their parts must be related, i.e. they must be designed to work as a coherent entity. The most important part of a human-designed system's study is probably the assessment, through the model representing it, of its performance. In fact, this could help the system's designer to make all the necessary modifications/improvements to the system's structure in order to increase its effectiveness.

The amount of information obtained by an action can be measured by the reduction of uncertainty resulting from this action. Accordingly a system's uncertainty is connected to its capacity in obtaining relevant information. Therefore a measure of uncertainty could be adopted as a measure of a system's effectiveness in solving related problems. In earlier papers we have used the total possibilistic uncertainty as well as the Shannon's entropy (total probabilistic uncertainty) - expressed in terms of the Dempster-Shafer mathematical theory of evidence for use in a fuzzy environment - for measuring the effectiveness of certain human designed systems in the areas of Education, of Artificial Intelligence and of Management (e.g. Problem Solving, Learning, Case-Based Reasoning, Decision Making, etc); see [8] and its references. Also Perdikaris ([9], [10]) has applied the same methods for measuring the student's geometrical reasoning skills with respect to the corresponding van Hieles' levels.

In this paper and in terms of the fuzzy model developed above we shall introduce another approach for measuring model building capacities (and hence CT capacities as well), known as the *'centroid method'*. According to this approach the centre of mass of the graph of the membership function involved provides an alternative measure of the system's performance. The application of the 'centroid method' in practice is simple and evident and, in contrast to the measures of uncertainty, needs no complicated calculations at the final step.

For this, given a fuzzy subset $A = \{(x, m(x)): x \in U\}$ of the universal set U of the discourse with membership function $m: U \to [0, 1]$, we correspond to each $x \in U$ an interval of values from a prefixed numerical distribution, which actually means that we replace U with a set of real intervals. Then, we construct the graph F of the membership function $y=m(x)$.There is a commonly used in fuzzy logic approach to measure performance with the pair of numbers (x_c, y_c) as the coordinates of the *centre of mass*, say F_c, of the graph F, which we can calculate using the following well-known [11] formulas:

$$x_c = \frac{\iint_F x\,dx\,dy}{\iint_F dx\,dy}, \quad y_c = \frac{\iint_F y\,dx\,dy}{\iint_F dx\,dy} \qquad (1)$$

Concerning the modelling process, when a student obtains a mark, say y, between 0 and 5, we characterize his/her performance as very low (a) if $y \in [0, 1)$, as low (b) if $y \in [1, 2)$, as intermediate (c) if $y \in [2, 3)$, as high (d) if $y \in [3, 4)$ and as very high (e) if $y \in [4,5]$ respectively. Therefore in this case the graph F of the corresponding fuzzy subset of U is the bar graph of Figure 1.

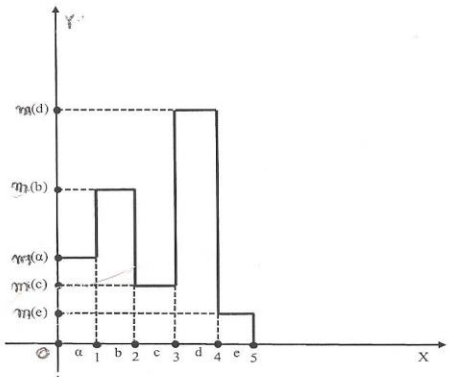

Figure 1: Bar graphical data representation

It is easy to check that, if the bar graph consists of n rectangles (in Figure 1 we have n=5), the formulas (1) can be reduced to the following formulas:

$$x_c = \frac{1}{2}\left(\frac{\sum_{i=1}^{n}(2i-1)y_i}{\sum_{i=1}^{n} y_i}\right), y_c = \frac{1}{2}\left(\frac{\sum_{i=1}^{n} {y_i}^2}{\sum_{i=1}^{n} y_i}\right) \quad (2).$$

Indeed, in this case $\iint_F dxdy$ is the total mass of the system which is equal to $\sum_{i=1}^{n} y_i$, $\iint_F xdxdy$ is the moment about the y-axis which is equal to $\sum_{i=1}^{n}\iint_{F_i} xdxdy = \sum_{i=1}^{n}\int_0^{y_i} dy \int_{i-1}^{i} xdx = \sum_{i=1}^{n} y_i \int_{i-1}^{i} xdx = \frac{1}{2}\sum_{i=1}^{n}(2i-1)y_i$, and $\iint_F ydxdy$ is the moment about the y-axis which is equal to $\sum_{i=1}^{n}\iint_{F_i} ydxdy = \sum_{i=1}^{n}\int_0^{y_i} ydy \int_{i-1}^{i} dx = \sum_{i=1}^{n}\iint_{F_i} xdxdy = \sum_{i=1}^{n}\int_0^{y_i} dy \int_{i-1}^{i} xdx = \sum_{i=1}^{n}\int_0^{y_i} ydy = \frac{1}{2}\sum_{i=1}^{n} {y_i}^2$.

From the above argument, where F_i, i=1,2,…,n , denote the n rectangles of the bar graph, it becomes evident that the transition from (1) to (2) is obtained under the assumption that all the intervals have length equal to 1 and that the first of them is the interval [0, 1].

In our case (n=5) formulas (2) are transformed into the following form:

$$x_c = \frac{1}{2}\left(\frac{y_1+3y_2+5y_3+7y_4+9y_5}{y_1+y_2+y_3+y_4+y_5}\right),$$

$$y_c = \frac{1}{2}\left(\frac{{y_1}^2+{y_2}^2+{y_3}^2+{y_4}^2+{y_5}^2}{y_1+y_2+y_3+y_4+y_5}\right). \quad (3)$$

Normalizing our fuzzy data by dividing each m(x), $x \in U$, with the sum of all membership degrees we can assume without loss of the generality that $y_1+y_2+y_3+y_4+y_5 = 1$. Therefore we can write:

$$x_c = \frac{1}{2}\left(y_1+3y_2+5y_3+7y_4+9y_5\right),$$

$$y_c = \frac{1}{2}\left({y_1}^2+{y_2}^2+{y_3}^2+{y_4}^2+{y_5}^2\right) \quad (4)$$

with $y_i = \frac{m(x_i)}{\sum_{x\in U} m(x)}$, where x_1 = a, x_2 =b, x_3= c, x_4 = d and x_5 = e.

But $0 \le (y_1-y_2)^2 = y_1^2+y_2^2-2y_1y_2$, therefore $y_1^2+y_2^2 \ge 2y_1y_2$,with the equality holding if, and only if, $y_1=y_2$.

In the same way one finds that $y_1^2+y_3^2 \geq 2y_1y_3$, and so on. Hence it is easy to check that $(y_1+y_2+y_3+y_4+y_5)^2 \leq 5(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2)$, with the equality holding if, and only if $y_1=y_2=y_3=y_4=y_5$.

But $y_1+y_2+y_3+y_4+y_5 =1$, therefore $1 \leq 5(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2)$ (5), with the equality holding if, and only if $y_1=y_2=y_3=y_4=y_5=\frac{1}{5}$.

Then the first of formulas (4) gives that $x_c = \frac{5}{2}$. Further, combining the inequality (5) with the second of formulas (4) one finds that $1 \leq 10y_c$, or $y_c \geq \frac{1}{10}$ Therefore the unique minimum for y_c corresponds to the centre of mass $F_m (\frac{5}{2}, \frac{1}{10})$.

The ideal case is when $y_1=y_2=y_3=y_4=0$ and $y_5=1$. Then from formulas (3) we get that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$.Therefore the centre of mass in this case is the point $F_i (\frac{9}{2}, \frac{1}{2})$.

On the other hand the worst case is when $y_1=1$ and $y_2=y_3=y_4= y_5=0$. Then for formulas (3) we find that the centre of mass is the point $F_w (\frac{1}{2}, \frac{1}{2})$.

Therefore the area where the centre of mass F_c lies is represented by the triangle F_w F_m F_i of Figure 2.

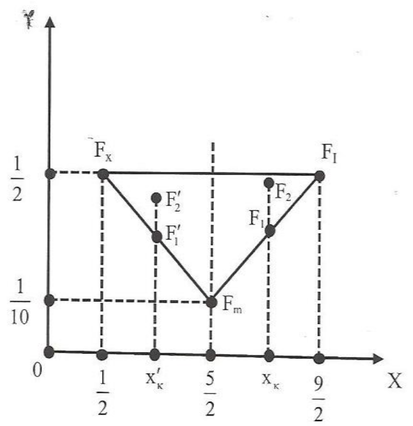

Figure 2: Graphical representation of the "area" of the centre of mass

Then from elementary geometric considerations it follows that for two groups of a system's objects with the same $x_c \geq 2,5$ the group having the centre of mass which is situated closer to F_i is the group with the higher y_c; and for two groups with the same $x_c <2.5$ the group having the centre of mass which is situated farther to F_w is the group with the lower y_c. Based on the above considerations it is logical to formulate our criterion for comparing the groups' performances in the following form:

- *Among two or more groups the group with the biggest x_c performs better.*
- *If two or more groups have the same $x_c \geq 2.5$, then the group with the higher y_c performs better.*
- *If two or more groups have the same $x_c < 2.5$, then the group with the lower y_c performs better.*

Notice that Subbotin et al., based on our fuzzy model for the process of learning [12], applied the "centroid" method on comparing students' mathematical learning abilities [13] and for measuring the scaffolding (assistance) effectiveness provided by the teacher to students [14]. Also, in a more recent paper, Voskoglou and Subbotin [15] applied this method in measuring the individuals' analogical reasoning skills.

4. Classroom Experiments

Exploratory investigations have demonstrated how exposure to CT enhances the way students approach modelling problems ([16], [17], [1], etc). In order to investigate further the above fact, but also to illustrate the use of our fuzzy model developed above in practice, we performed recently the following two experiments at the Graduate Technological Educational Institute (TEI) of Patras, in Greece.

In the first experiment our subjects were 35 students of the School of Technological Applications, i.e. future engineers, being at their second term of studies. Notice that part (about the 1/3) of the lectures and the exercises of mathematical courses for the students of this School are performed in a computer laboratory, where the instructor presents the corresponding mathematical topics in a more "live" and attractive to students' way, while the students themselves, divided in small groups, use already existing mathematical software to solve the problems with the help of computers.

Our basic tool in this experiment was a list of 10 problems involving mathematical modelling (MM) given to them for solution (time allowed 3 hours). The mathematical topics related to these problems included elementary and linear algebra, differential and integral calculus, elementary differential equations and probability theory (see Appendix).

Before starting the experiment we gave the proper instructions to students emphasizing, among the others, that we were interested for <u>all their efforts</u> (successful or not) during the MM process, and therefore they must keep records on their papers for all of them, at <u>all stages</u> of the MM process. This manipulation enabled as in obtaining realistic data from our experiment for each stage of the MM process and not only those based on students' final results that could be obtained in the usual way of graduating their papers.

Examining students' papers by using the criterion applied above for the construction of Figure 1 we found that 15, 12 and 8 students had intermediate, high and very success respectively at stage S_1 of analysis/mathematization. Therefore we obtained that $n_{1a}=n_{1b}=0$, $n_{1c}=15$, $n_{1d}=12$ and $n_{1e}=8$. Thus, by the definition of the corresponding membership function given in the second section, S_1 is represented by a fuzzy subset of U of the form:

$$A_1 = \{(a,0),(b,0),(c, 0.5),(d, 0.25),(e,0..25).$$

In the same way we represented the stages S_2 and S_3 as fuzzy sets in *U* by

$$A_2 = \{(a,0),(b,0),(c, 0.5),(d, 0.25),(e,0)\},$$
$$A_3 = \{(a, 0.25),(b, 0.25),(c, 0.25),(d,0),(e,0)\}$$

respectively.

Next we calculated the membership degrees of the 5^3 (ordered samples with replacement of 3 objects taken from 5) in total possible students' profiles as it is described in the second section (see column of $m_s(1)$ in Table 1). For example, for the profile *s=(c, c, a)* one finds that

$$m_s = m_{A_1}(c).\ m_{A_2}(c).\ m_{A_3}(a) = 0.5 \times 0 .5 \times 0.25 = 0.06225.$$

Further, from the values of the column of $m_s(1)$ it turns out that the maximal membership degree of students' profiles is *0.06225*. Therefore the possibility of each *s* in U^3 is given by

$$r_s = \frac{m_s}{0.06225}.$$

One, in order to be able to make the corresponding comparisons, could also calculate the probabilities of the students' profiles using the formula for p_s given in section 2. However notice that, according to Shackle [18] and many others after him, human reasoning is better presented by possibility rather than by probability theory. Therefore, adopting the above view, we considered that the calculation of the probabilities is not necessary.

Table 1: Profiles with non zero membership degrees

A_1	A_2	A_3	$m_s(1)$	$r_s(1)$	$m_s(2)$	$r_s(2)$	$f(s)$	$r(s)$
B	B	b	0	0	0.016	0.258	0.016	0.129
B	B	a	0	0	0.016	0.258	0.016	0.129
B	A	a	0	0	0.016	0.258	0.016	0.129
C	c	c	0.062	1	0.062	1	0.124	1
C	c	a	0.062	1	0.062	1	0.124	1
C	c	b	0	0	0.031	0.5	0.031	0.25
C	a	a	0	0	0.031	0.5	0.031	0.25
C	b	a	0	0	0.031	0.5	0.031	0.25
C	b	b	0	0	0.031	0.5	0.031	0.25
D	d	a	0.016	0.258	0	0	0.016	0.129
D	d	b	0.016	0.258	0	0	0.016	0.129
D	d	c	0.016	0.258	0	0	0.016	0.129
D	a	a	0	0	0.016	0.258	0.016	0.129
D	b	a	0	0	0.016	0.258	0.016	0.129
D	b	b	0	0	0.016	0.258	0.016	0.129
D	c	a	0.031	0.5	0.031	0.5	0.062	0.5
D	c	b	0.031	0.5	0.031	0.5	0.062	0.5
D	c	c	0.031	0.5	0.031	0.5	0.062	0.5
E	c	a	0.031	0.5	0	0	0.031	0.25
E	c	b	0.031	0.5	0	0	0.031	0.25
E	c	c	0.031	0.5	0	0	0.031	0.25
E	d	a	0.016	0.258	0	0	0.016	0.129
E	d	b	0.016	0.258	0	0	0.016	0.129
E	d	c	0.016	0.258	0	0	0.016	0.129

(The outcomes of Table 1 were obtained with accuracy up to the third decimal point).

A few days later we performed the same experiment with a group of 50 students from the School of Management and Economics being also at their second term of studies. The topics covered in the mathematics course of the first term were almost the same with the students of the first group from the School of Technological Applications. Further, according to the marks obtained in this course, the two groups were equivalent. The only difference was that the lectures in the mathematical courses for the students of the School of Management and Economics are performed in the classical way on the board including a number of exercises and examples connecting mathematics with real world applications and problems. The students participate in solving these problems.

Working as in the first experiment we found that

$$A_1=\{(a, 0),(b, 0.25),(c, 0.5),(d, 0.25),(e, 0)\},$$

$$A_2=\{(a, 0.25),(b, 0.25),(c, 0.5),(d, 0),(e, 0)\}$$

$$A_3=\{(a, 0.25),(b, 0.25),(c, 0.25),(d, 0),(e, 0)\}.$$

Then we calculated the membership degrees of all possible profiles of the student group (column of m_s (2) in Table 1). Further, since the maximal membership degree is again 0.06225, the possibility of each *s* is given by the same formula as for the first group. The values of the possibilities of all profiles are given in column of $r_s(2)$ of Table 1.

Finally, in order to study the combined results of the two groups' performance we calculated the pseudo-frequencies $f(s) = m_s(1)+m_s(2)$ and the combined possibilities of all profiles (see the last two columns of Table 1) as it has been described in section 2 of the present paper.

Next, in order to compare the two groups' performance by the 'centroid method', let us denote by A_{ij} the fuzzy subset of U attached to the stage S_j , j=1,2,3 , of the MM process with respect to the student group i, i=1,2.

At the first stage of analysis/mathematization we have

$$A_{11} = \{(a, 0),(b, 0),(c, 0.5),(d, 0.25),(e, 0.25), \quad A_{21}= \{(a, 0),(b, 0.25),(c, 0.5),(d , 0.25),(e, 0)\}$$

and respectively

$$x_{c11} = \frac{1}{2}(5 \text{ x } 0.5 + 7 \text{ x } 0.25 + 9 \text{ x } 0.25) = 3.25, \quad x_{c21} = \frac{1}{2}(3 \text{ x } 0.25 + 5 \text{ x } 0.5 + 7 \text{ x } 0.25) = 2.25$$

Thus, by our criterion the first group demonstrates better performance.
At the second stage of solution we have:

$$A_{12} = \{(a, 0),(b, 0),(c, 0.5),(d, 0.25),(e, 0)\}, \quad A_{22}=\{(a, 0.25),(b, 0.25),(c, 0.5),(d, 0),(e, 0)\}.$$

Normalizing the membership degrees in the first of the above fuzzy subsets of U (0.5 : 0,.75 $\approx$ 0.67 and 0.25 : 0.75 $\approx$ 0.33) we get

A_{12} = {(a, 0),(b, 0),(c, 0.67),(d, 0.33),(e, 0)}, A_{22}={(a, 0.25),(b, 0.25),(c, 0.5),(d, 0),(e, 0)}

and respectively

$$x_{c12} = \frac{1}{2}(5 \text{ x } 0.67 + 7 \text{ x } 0.33) = 2.83,\ x_{c22} = \frac{1}{2}(0.25 + 3 \text{ x } 0.25 + 5 \text{ x } 0.25) = 1.125$$

By our criterion, the first group again demonstrates a significantly better performance. Finally, at the third stage of validation/implementation we have

$$A_{13} = A_{23} = \{(a, 0.25),(b, 0.25),(c, 0.25),(d, 0),(e, 0)\},$$

which obviously means that at this stage the performances of both groups are identical.

Based on our calculations we can conclude that the first group demonstrated a significantly better performance at the stages of analysis/mathematization and of solution, but performed identically with the second one at the stage of validation/implementation.

Thus, the results of our experiments give a strong indication that the use of computers as a tool for PS (students of the first group) enhances the students' abilities in solving real world problems involving mathematical modelling.

5. Conclusions

The following conclusions can be drawn from the discussion performed in this paper:

- Computational thinking (CT) is a new problem solving method named for its extensive use of computer science techniques. It synthesizes critical thinking and existing knowledge and applies them to solve complex real world technological problems.
- Modelling thinking constitutes the essence of CT, since it synthesises all the other components of CT (abstract, logical and constructive thinking) for the solution of the corresponding problem.
- In this paper we developed a fuzzy model for the CT process by representing the main stages of the modelling process as fuzzy subsets of a set of linguistic labels characterizing the modellers' performance in each of these stages. We also applied the 'centroid' method in obtaining a measure of the individuals' CT skills.
- Two classroom experiments were presented illustrating the use of our fuzzy model in practice. The results of these experiments give a strong indication that the use of computers as a tool for problem solving enhances the students' abilities in solving real world problems involving mathematical modelling. This is also crossed by us and by other researchers in earlier papers.

References

[1] Voskoglou, M. Gr. & Buckley, S., "Problem Solving and Computers in a Learning Environment ", *Egyptian Computer Science Journal*, 36(2), 2012 (in press).

[2] Liu, J. & Wang, L., "Computational Thinking in Discrete Mathematics", *IEEE 2nd International Workshop on Education Technology and Computer Science*, 413-416, 2010.

[3] Taha, H. A., "Operations Research – An Introduction", Second Edition, Collier Macmillan, N.Y., London, 1967.

[4] Haines, C.R. & Crouch, R., "Remarks on modeling cycle and interpretation of behaviours", in: Lesh, R.A., Galbraith, P. L., Haines, C.R. & Harford, A. (Eds.), *Modelling Students' Mathematical Modelling Competencies , (ICTMA13)*, Springer, NY, London, 145-154, 2010.

[5] Borroneo Ferri, R., "Modelling problems from a cognitive perspective". In: Haines, C.R., Galbraith,P., Bloom, W., Khan, S. (Eds.), *Mathematical Modelling: Education, Engineering and Economics,(ICTMA 12)*, Horwood Publishing, Chichester, 260-270, 2007.

[6] Doer, H.M., "What knowledge do teachers need for teaching mathematics through applications and modeling? ". In: Blum, W., Galbraith, P.L., Henn, H-W. and Niss, M. (Eds.), *Modelling and Applications in Mathematics Education*, Springer, NY, 69-78, 2007.

[7] Klir, G. J. & Folger, T. A., "Fuzzy Sets, Uncertainty and Information", Prentice-Hall, London, 1988.

[8] Voskoglou, M. Gr., "Stochastic and fuzzy models in Mathematics Education, Artificial Intelligence and Management", Lambert Academic Publishing, Saarbrucken, Germany, 2011 (for more details look at http://amzn.com./3846528218).

[9] Perdikaris, S., "Measuring the student group capacity for obtaining geometric information in the van Hiele development thought process: A fuzzy approach", Fuzzy Sets and Mathematics, 16 (3), 81-86, 2002.

[10] Perdikaris, S., "Using Fuzzy Sets to Determine the Continuity of the van Hiele Levels", Journal of Mathematical Sciences & Mathematics Education, 6(1), 39-46, 2011.

[11] van Broekhoven, E. & De Baets, B., "Fast and accurate centre of gravity defuzzification of fuzzy system outputs defined on trapezoidal fuzzy partitions", Fuzzy Sets and Systems, 157, Issue 7, 904-918, 2006.

[12] Voskoglou, M. Gr., "The process of learning mathematics: A fuzzy set approach", Heuristics and Didactics of Exact Sciences (Ukraine), 10, 9-13, 1999.

[13] Subbotin, I., Badkoobehi, H. & Bilotskii, N., "Application of Fuzzy Logic to Learning Assessment", Didactics of Mathematics: Problems and Investigations. Volume 22, 38-41, 2004.

[14] Subbotin, I., Mossovar-Rahmani, F. & Bilotskii, N., "Fuzzy logic and the concept of the Zone of Proximate Development", Didactics of Mathematics: Problems and Investigations, Volume 36, 101- 108, 2011.

[15] Voskoglou, M. Gr. & Subbotin, I. Ya., "Fuzzy Models for Analogical Reasoning", International Journal of Applications of Fuzzy Sets, 2, 19-38, 2012.

[16] Lewandowski, G., Bouvier, D., McCartney, R., Sanders, K. & Simon, B., "Common sense computing (episode 3): Concurrency and concert tickets", *Proceedings of the Third International Workshop on Computing Education Research (ICER '07),* 2007.

[17] Weller, K. et al., "Student performance and attitudes in courses based on APOS theory and the ACE teaching style". In A. Selden et al. (Eds.), *Research in collegiate mathematics education V*, 97-131, Providence, RI: American Mathematical Society, 2003.

[18] Shackle, G. L. S., "Decision, Order and Time in Human Affairs", Cambridge University Press, Cambridge, 1961

Appendix

The problems given for solution to the students in our classroom experiments:

Problem 1: We want to construct a channel to run water by folding the two edges of an orthogonal metallic leaf having sides of length 20cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how we can run the maximum possible quantity of the water?

Remark: The correct solution is obtained by folding the edges of the longer side of the leaf. Some students solved the problem by folding the edges of the other side and failed to realize (validation of the model) that their solution was wrong.

Problem 2: A car dealer has a mean annual demand of 250 cars, while he receives 30 new cars per month. The annual cost of storing a car is 100 euros and each time he makes a new order he pays an extra amount of 2200 euros for general expenses (transportation, insurance etc). The first cars of a new order arrive at the time when the last car of the previous order has been sold. How many cars must he order in order to achieve the minimum total cost?

Problem 3: An importation company codes the messages for the arrivals of its orders in terms of characters consisting of a combination of the binary elements 0 and 1. If it is known that the arrival of a certain order will take place from 1st until the 16^{th} of March, find the minimal number of the binary elements of each character required for coding this message.

Problem 4: Let us correspond to each letter the number showing its order into the alphabet (A=1, B=2, C=3 etc). Let us correspond also to each word consisting of 4 letters a 2X2 matrix in the obvious way; e.g. the matrix $\begin{bmatrix} 19 & 15 \\ 13 & 5 \end{bmatrix}$ corresponds to the word SOME. Using the matrix $E=\begin{bmatrix} 8 & 5 \\ 11 & 7 \end{bmatrix}$ as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how he (she) could decode your message?

Problem 5: The demand function $P(Q_d)=25-Q_d^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s)=2Q_s+1$ represents the prices at which different quantities Q_s of the same good will be supplied. If the market's equilibrium occurs at (Q_0,P_0), the producers who would supply at lower price than P_0 benefit. Find the total gain to producers'.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corresponding ball to the box before the next lottery. Find the probability of getting all the balls that he draws out of the box different.

Problem 7: A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour?

Problem 8: The population of a country is increased proportionally. If the population is doubled in 50 years, in how many years it will be tripled?

Problem 9: A wine producer has a stock of wine greater than 500 and less than 750 kilos. He has calculated that, if he had the double quantity of wine and transferred it to bottles of 12, 25, or 40 kilos, it would be left over 6 kilos each time. Find the quantity of stock.

Problem 10: Among all cylindrical towers having a total surface of 180π m^2, which one has the maximal volume?

Remark: Some students didn't include to the total surface the one base (ground-floor) and they found another solution, while some others didn't include both bases (roof and ground-floor) and they found no solution, since we cannot construct a cylinder with maximal volume from its surrounding surface.