Egyptian Computer Science Journal Vol. 38 No.3 September 2014 ISSN-1110-2586

Probability and Fuzziness in Decision Making

Michael Gr. Voskoglou

School of Technological Applications
Graduate Technological Educational Institute of Patras, GREECE
mvosk@hol.gr , http://eclass.teipat.gr/eclass/courses/523102

Abstract

In the present paper we develop a stochastic (Markov chain) model for a mathematical description of the decision making (DM) process and we apply principles of fuzzy logic (FL) in representing the DM process in a fuzzy environment. Further, we develop an innovative approach for studying the step of verification of a chosen decision, which is based on principles of FL (a special form of the COG defuzzification technique) and we compare this approach with other traditional (statistical) approaches that can be applied for the same reason. Examples are also given to illustrate the use of our results in practice.

Keywords: *Decision making (DM), GPA Method, Markov chains (MC), Fuzzy sets, Fuzzy DM, Centre of gravity (COG) defuzzification technique.*

1. Introduction

Decision Making (DM) is the process of choosing a solution between two or more alternatives, aiming to achieve the best possible results for a given problem. Obviously the above process has sense if, and only if, there exist more than one *feasible solutions*, together with a suitable criterion (or criteria) that helps the decision maker (d-m) to choose the best among these solutions. We recall that a solution is characterized as feasible, if it satisfies all the restrictions imposed onto the real system by the statement of the problem as well as all natural restrictions imposed onto the problem by the real system, e.g. if x denotes the quantity of stock of a product, it must be $x \geq 0$. The choice of the suitable criterion, especially when the results of DM are affected by random events, depends upon the desired goals of the d-m; e.g. optimistic or conservative criterion etc).

The rapid technological progress , the impressive development of the transport means, the globalization of our modern society, the enormous changes happened to the local and international economies and other similar reasons led during the last 50-60 years to a continuously increasing complexity of the problems of our everyday life. As a result the DM process became in many cases a very difficult task, so that it is impossible to be based on the d-m's experience, intuition and skills only, as it usually happened in the past. Thus, from the beginning of the 1950's a progressive development started of a systematic methodology for the DM process, which is based on Probability Theory, Statistics, Economics, Psychology, etc and it is known as *Statistical Decision Theory* (e.g. see [1]).

According to the nowadays existing standards the DM process involves the following steps:

- **d_1:** *Analysis* of the decision-problem, i.e. understanding, simplifying and reformulating the problem in a way permitting the application of the principles of SDT on it.
- **d_2:** Collection from the real system and interpretation of all the necessary information related to the problem.
- **d_3:** Determination of all the alternative feasible solutions.
- **d_4:** Choice of the best solution in terms of the suitable (according to the decision maker's goals and targets) criterion.

One could add one more step to the DM process, the *verification of the chosen decision* according to the results obtained by applying it in practice. However, this step is extended to areas, which due to their depth and importance for the administrative rationalism have become autonomous. Therefore, it is usually examined separately from the other steps of the DM process.

Notice that the first three steps of the DM process presented above are *continuous* in the sense that the completion of each one of them usually needs some time, during which the decision maker's reasoning is characterized by transitions between hierarchically neighbouring steps. In other words *the DM process is not a linear process*. Accordingly its flow-diagram is represented in Figure 1 below:

$$\mathbf{d_1} \leftrightarrow \mathbf{d_2} \leftrightarrow \mathbf{d_3} \rightarrow \mathbf{d_4}$$

Figure 1: The flow-diagram of the DM process

At this point it is worthy to notice that frequently in our everyday life a DM problem is expressed in an ambiguous way involving a degree of uncertainty. For example, this happens when a company wishes to employ as a sales manager the candidate with the best qualifications, provided that his/her request for salary is not very high and that his/her residence is not very far from the companies place (see section 3). In such cases the classical Statistical Decision Theory based on *Probability* and on principles of the traditional bivalent logic (yes-no) is proved inadequate for helping the d-m to take the correct decision. On the contrary, *Fuzzy Logic* (FL), based on the notion of fuzzy sets introduced by Zadeh [2] in 1965, due to its nature of including multiple values, offers a rich field of resources for this purpose.

In the present paper we are going to present a stochastic (Markov chain) model for a mathematical description of the DM process and to apply principles of FL in representing the DM process in a fuzzy environment. Accordingly the rest of the paper is organized as follows: In section 2 we develop our Markov model for the DM process and we present an example illustrating its use in practice. In section 3 we present examples illustrating the process of DM under fuzzy conditions. Further, in section 4 we develop an innovative approach for studying the step of verification of a decision, which is based on principles of FL and we compare this approach with other traditional statistical approaches. Finally, our last section 5 is devoted to the statement of our conclusions and to discussion of our future plans for further research on the subject.

2. The Markov Chain Model

Roughly speaking, a *Markov chain* (MC) is a special type of a stochastic process that moves in a sequence of steps (phases) through a set of states and whose main characteristic, known as the *Markov property*, is that it has memory of only one state. This means that the probability of entering a certain state at a certain step of the process depends only on the state occupied in the previous step. However, in practice during the process of mathematical modelling there is usually a need to simplify the real system in a way that enables the formulation of it to a form ready for mathematical treatment (*assumed real system*, e.g. see [3]. This enables a number of authors to state the Markov property in a more general context by accepting that the probability of entering a certain state at a certain step of the process, although it is not necessarily independent from older states, it depends *mainly* on the state occupied in the previous step (e.g. [4], Chapter 12). When a MC has a finite number of states, it is called a *finite Markov chain.* Here, we assume the reader to be familiar with the basics of the theory of finite MC, for which we refer to the book [5].

In obtaining a mathematical formulation of the DM process we introduce a finite MC on its steps by adopting the notion of the Markov property in its wider context (see above), a fact which is very close to the reality. In other words, this means that the states of our chain are the steps d_i, i = 1, 2, 3, 4, of the DM process introduced in the previous section. It is logical to accept that d_1 is always the starting state. Further, we observe that, when the chain reaches the state d_4 (end of the DM process) it is impossible to leave it. This means that d_4 is the unique *absorbing state* of the chain. Therefore, since it is obviously possible from any state to reach the absorbing state d_4, not necessarily in one step (see Figure 1), our chain is an *absorbing Markov chain* ([5], Chapter III).

Let us now denote by p_{ij} the *transition probability* from state d_i to d_j , i,j = 1,2,3,4 . Then the *transition matrix* of our MC is

$$A = \begin{array}{c} \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{array}\begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_3 & d_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ p_{21} & 0 & p_{23} & 0 \\ 0 & p_{32} & 0 & p_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

, where we obviously have (probability of a certain fact) that

$$p_{21}+p_{23} = p_{32}+p_{34} = 1 \qquad (1)$$

Let us also denote by $\varphi_0, \varphi_1, \varphi_2$, ... the several steps of the chain and let

$P_i= [p_1^{(i)}\ p_2^{(i)}\ p_3^{(i)}\ p_4^{(i)}]$ be the row - matrix giving the probabilities for the chain to be in each one of its states at the step φ_i , i = 0,1,2,.... . Then, since d_1 is always the starting state, we have that $P_0 = [1\ 0\ 0\ 0]$. Further it is well known that $P_{i+1} = P_i A$ and therefore

$P_1 = P_0A = [0\ 1\ 0\ 0]$

$P_2 = P_1 A = [p_{21}\ 0\ p_{23}\ 0]$

$P_3 = P_2 A = [0\ \ p_{21}+ p_{23}p_{32}\ \ 0\ \ p_{23}p_{34}]$ (2).

$P_4 = P_3A = [p_{21}^2+p_{21}p_{23}p_{32}\ \ 0\ \ p_{21}p_{23}+ p_{23}^2 p_{32}\ \ p_{23}p_{34}]$ and so on.

In general an inductive argument shows that $P_n = P_0A^n$, n = 1, 2, 3,... .

We shall now bring A to its *standard form* A* by listing the absorbing state first and then we partition A* as follows:

$$A^* = \begin{array}{c|c} & \begin{matrix} d_4 & d_1 & d_2 & d_3 \end{matrix} \\ \begin{matrix} d_4 \\ d_1 \\ d_2 \\ d_3 \end{matrix} & \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & p_{21} & 0 & p_{23} \\ p_{34} & 0 & p_{32} & 0 \end{array}\right] \end{array}$$

. Let $Q = \begin{bmatrix} 0 & 1 & 0 \\ p_{21} & 0 & p_{23} \\ 0 & p_{32} & 0 \end{bmatrix}$ be the matrix of non absorbing states and denote by I_3 the 3x3 unitary matrix , then it is well known that I_3-Q is always an invertible matrix (e.g. see [6], section 2).

The *fundamental matrix* N of the chain is defined by $N= (I_3-Q)^{-1} = \frac{1}{D(I_3-Q)}$ adj $(I_3$-Q), where $D(I_3$-Q) denotes the determinant and adj $(I_3$-Q) denotes the adjoint matrix of I_3-Q. We recall that the elements of adj $(I_3$-Q) are the algebraic complements of the elements of the transpose matrix of I_3-Q. Thus, by a straightforward calculation and using relation (1) we obtain that

$$N = \frac{1}{p_{23}p_{34}} \begin{bmatrix} 1-p_{32}p_{23} & 1 & p_{23} \\ p_{21} & 1 & p_{23} \\ p_{21}p_{32} & p_{32} & p_{23} \end{bmatrix} = [n_{ij}]\ ,\ i,j = 1,2,3 \qquad (3)$$

We recall that *the ij-th entry of N gives the mean number of times in state d_j before the absorption, when the chain is started in state d_i*. Therefore, since in our case d_1 is always the starting state, the mean number of steps taken before absorption is given by:

$$t = \sum_{i=1}^{3} n_{1i} = \frac{2+p_{23}p_{34}}{p_{23}p_{34}} \qquad (4).$$

Obviously, the bigger the value of t, the more the difficulties that a d-m faces during the DM process; in other words t provides an indication for the difficulty of the DM process (other indications for the difficulty of the DM process could be the time spent by the d-m to complete it, etc).

The following example illustrates the use of the above MC model in practice:

EXAMPLE: In the province of Akhaia of Greece the manager of a local company A, which produces, pucks up and trades olive oil, has employed a specialist to help him in deciding about the proper place for building a new factory. The deal is to pay the specialist for six working hours (w. h.) whenever an analysis of the DM problem is required, for 54 w. h. whenever collection and interpretation of the necessary information is needed, for 28 w. h. whenever the determination of all feasible solutions is attempted and for 9 w. h. for the final choice of the best decision. The manager wants to determine the probability for the DM process to be terminated in four steps and to estimate the mean number of steps needed before taking the decision as well as the expected number of w. h. to be paid to the specialist for his services.

We shall analyze the DM process for the above DM problem according to the lines of the above presented MC model:

d_1: Analysis of the DM problem

The analysis of the DM problem, performed by the specialist, showed that the profitability of the decision to be taken depends upon the types (qualities) of the oil produced by the existing in the area where A acts competitive companies.

d_2: Collection and interpretation of the necessary information

The relevant investigation has shown that there is only one competitive company in the area, say B, which produces three different types of oil, say W_1, W_2 and W_3.

d_3: Determination of the feasible solutions

The general situation of the area (communications, traffic, the already existing factories and storehouses of the companies A and B etc), combined with the funds available by the company A for the construction of the new factory, suggest that there are four favourable places , say P_1 , P_2 , P_3 and P_4 for the possible construction of the new factory. However, the need of some new information (data of the market's research) became necessary for the specialist at this point in order to be able to proceed to the choice of the best solution.

$d_3 \rightarrow d_2$: Going back from d_3 to d_2

The market's research has shown that the expected net profits of the company A with respect to the favourable places for the construction of the new factory and the types of the oil produced by the company B are those shown in Table 1 below:

Table1: Net profits of the company A

$$\begin{array}{c c} & \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ \begin{array}{c} W_1 \\ W_2 \\ W_3 \end{array} & \begin{bmatrix} 3 & 8 & 5 & 4 \\ 4 & 2 & 6 & 5 \\ 2 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

$d_2 \rightarrow d_3$: New transition from d_2 to d_3

From Table 1 it becomes evident that the feasible solution P_4 is worse than P_3 and therefore P_4 is rejected.

d_4: Choice of the best solution

The manager of the company does not want to risk for earning low profits by constructing the new factory, which means that the specialist must adopt a conservative criterion for the choice of the best place for building it. In such cases the most frequently used criterion is *the maximin of payoffs* (profits), due to *Wald*. The Wald's criterion, based on the

Murphy's law assuming that the worst possible fact to be happen will finally happen, suggests to maximize the minimal possible for each case profits. In other words, since the minimal expected profit from the choice of P_1 is 2 monetary units and the minimal profit from the choice of P_2 or of P_3 is 1 monetary unit (see Table 1), according to the Wald's criterion the place P_1 must be chosen for building the new factory.

Data evaluation

From the above analysis of the DM process it becomes evident that $p_{21} = 0$ and $p_{23} = 1$. We also claim that $p_{32} = p_{34} = 0.5$. In fact, when the MC reaches the state d_3 for first time, the probability of returning to d_2 at the next step is 1, since collection and interpretation of new information becomes necessary. Further, the second time that the MC reaches d_3 the probability of returning to d_2 at the next step is 0, since no more information is needed for the choice of the best solution. Therefore the transition probability p_{32} is equal to the mean $\frac{0+1}{2}$ and also $p_{34} = 1 - p_{32} = 0.5$.

Replacing the above values of the transition probabilities to the third of relations (2) we find that $P_3 = [0 \;\; 0.5 \;\; 0 \;\; 0.5]$, i.e. $p_4^{(3)} = 0.5$, which means that the probability for the DM process to be terminated in 4 steps is 50%. This could happen, if there was no feasible solution worse than one of the others and therefore we didn't reject any of them, as we did above for P_4.

Further, from relation (3) we obtain that $N = \frac{1}{0.5}\begin{bmatrix} 0.5 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0.5 & 1 \end{bmatrix}$, wherefrom we find that $n_{11} = 1$ and $n_{12} = n_{13} = 2$. Thus, the mean number of steps for the DM process before taking the decision is t = 5 steps, while the expected number of w. h. to be paid to the specialist is equal to 1*6 + 2*54 + 2*28 + 1*9 = 179 w. h.

3. DM under Fuzzy Conditions

In many cases a DM problem is expressed in an ambiguous way involving a degree of uncertainty. In such cases, as said in our introduction, while the classical Statistical Decision Theory cannot offer an effective help for the DM process, FL due to its nature of including multiple values, offers a rich field of resources. For general facts on FL we refer to the book [7]. The following example illustrates the standard process of DM under fuzzy conditions:

EXAMPLE 1: A company wants to employ as a sales manager the candidate with the best qualifications, provided that his/her request for salary is not very high and that his/her residence is in a close driving distance from the companies place. They are four candidates for the above position, say A, B, C and D with annual salary requests 29050, 25000, 14050, and 6250 euros respectively. Who of them is the best choice for the company under the above (fuzzy) conditions?

In this DM problem we have the *fuzzy goal (G)* of employing the candidate with the best qualifications under the *fuzzy constraints* that his request for salary must not be very high (C_1)

and that his/her residence must be in a close driving distance from the companies place (C_2). The steps of the DM process in such fuzzy situations are the following:

Step 1: Choice of the universal set of the discourse

In our case we must obviously consider as universal set the set U = {A, B, C, D} of the four candidates.

Step 2: Fuzzification of the decision problem's data

In this step the fuzzy goal and the fuzzy constraints of the problem are expressed as *fuzzy sets in U*. For this, we must define properly the corresponding for each case *membership function*.

For example, the membership function $m_{C_1}: U \rightarrow [0,1]$ for the fuzzy constraint C_1 can be defined by: $m_{C_1}(x)$ =1 for $s(x) < 6000$, $m_{C_1}(x) = 1 - 2 * 10^{-5} * s(x)\}$ for $6000 \leq s(x) \leq 30000$ and $m_{C_1}(x) = 0$ for $s(x) > 30000$, where $s(x)$ denotes the salary of the candidate x, for all x in U. Then $m_{C_1}(\mathrm{A}) = 1 - 2 * 0.2905 = 0.419$. Similarly we calculate the *membership degrees* of B, C and D and we write the constraint C_1 as a fuzzy set in U in the form of the symbolic sum C_1 = 0.419/A + 0.5/B + 0.719/C + 0.875/D.

In the same way (the relevant details are omitted here for reasons of brevity) we expressed the fuzzy goal G and the other fuzzy constraint C_2 as fuzzy sets in U in the form G = 0.9/A + 0.6/B + 0.8/C + 0.6/D and C_2 = 0.1/A + 0.9/B + 0.7/C + 1/D respectively[†].

Step 3: Evaluation of the fuzzy data

According to the *Bellman-Zadeh'sb criterion* for DM in a fuzzy environment [8], the *fuzzy decision F* expressed as a fuzzy set in U is the intersection of the fuzzy sets G , C_1 and C_2 of U and the solution of the problem corresponds to the element x of U having the highest membership degree in F.

Further, it is well known that the membership function of the intersection $G \cap C_1 \cap C_2$ is defined by $m_{G \cap C1 \cap C_2}(x) = m_F = \min\{m_G(x), m_{C_1}(x), m_{C_2}(x)\}$ for all x in U. Therefore it is easy to check that F = 0.1/A + 0.5/B + 0.7/C + 0.6/D.

Step 4: Defuzzification

The highest membership degree in F is 0.7 and corresponds to the candidate C. Therefore the candidate C is the best choice for the company.

The fuzzy model of Bellman-Zadeh can be further extended to accommodate the relative importance that could exist for the goal and constraints by using *weighting coefficients*. The following example illustrates this case:

[†] [†]We recall that the definition of the membership function is more or less arbitrary usually depending on statistical data collected from the population that we study. However a necessary condition for the creditability of the fuzzy model in representing the corresponding real situation is that the choice of the membership function is compatible with the rules of the common logic.

EXAMPLE 2: Reconsider Example 1 and assume that the Management Council of the company, taking into account the existing company's budget, the results of the oral interviews of the four candidates and some other relevant factors, decided to attach weights 0.5, 0.2 and 0.3 to the goal G and to the constraints C_1 and C_2 respectively. Which will be the company's choice under these conditions?

In this case the membership function of the fuzzy decision F is defined through a linear combination of the weighted goal and constraints of the form $m_F(x) = w_1 * m_G(x) + w_2 * m_{C_1}(x) + w_3 * m_{C_2}(x)$, where $m_G(x)$, $m_{C_1}(x)$, $m_{C_2}(x)$ are the membership degrees in G, C_1 and C_2 respectively of each x in U (see Example 1) and the coefficients w_1, w_2 and w_3 are the weights attached to the fuzzy goal and constraints respectively, with $w_1 + w_2 + w_3 = 1$ ([t], Chapter 6). Therefore the membership degree of the candidate A in the fuzzy decision F in this case is $m_F(A) = 0.5 * 0.9 + 0.419 * 0.2 + 0.1 * 0.3 = 0.638$. In the same way we find that $m_F(B) = 0.67$, $m_F(C) = 0.7538$ and $m_F(D) = 0.775$. Therefore the candidate D will be the company's choice in this case.

4. Verification of a Decision: A Fuzzy Approach

As it has been explained in our introduction, the verification of a taken decision is a step of the DM process, which is usually examined separately from its other steps. In this section we shall apply an innovative approach for examining this important step, which is based on principles of FL. More explicitly, we shall apply a widely used in FL defuzzification method, usually referred as the *center of gravity* (COG), or as the *centroid technique*. According to the COG method the defuzzification of a fuzzy situation's data is succeeded through the calculation of the coordinates of the COG of the level's section contained between the graph of the membership function associated with this situation and the OX axis (e.g. see [9]).

Prof. Subbotin (State University, Los Angeles, USA) and the author of this article have adapted properly several times in the past the COG technique (either collaborating or independently to each other) for assessing students' skills in a number of different (mainly mathematical) tasks (e.g. [10-11], [13-16], etc), for testing the effectiveness of a CBR system [12] and for measuring Bridge players' performance [17]. Here, using similar techniques, we shall adapt the COG method for examining the step of verification of a taken decision and we shall compare this approach with other traditional (statistical) approaches that can be applied for the same reason. All the above are illustrated by the following example:

EXAMPLE: A car industry decided to circulate its new model in the market in two different types, the luxury (L) Class and the regular (R) Class. Six months after the purchase of their cars the customers were asked to complete a written questionnaire concerning the degree of satisfaction for their new cars. Their answers were marked by the industry's marketing department within a climax from 0 to 100 and they were divided in the following five categories according to the corresponding scores: A (90-100) = Full satisfied customers, B (75-89) = Very satisfied customers, C (60-74) = Satisfied customers, D (50-59) = Rather satisfied customers and E (0-49) = Unsatisfied customers.

The scores of the customers' answers were the following:

L Class: 100(5 times), 99(3), 98(10), 95(15), 94(12), 93(1), 92 (8), 90(6), 89(3), 88(7), 85(13), 82(4), 80(6), 79(1), 78(1), 76(2), 75(3), 74(3), 73(1), 72(5), 70(4), 68(2), 63(2), 60(3), 59(5), 58(1), 57(2), 56(3), 55(4), 54(2), 53(1), 52(2), 51(2), 50(8), 48(7), 45(8), 42(1), 40(3), 35(1).

R Class: 100(7), 99(2), 98(3), 97(9), 95(18), 92(11), 91(4), 90(6), 88(12), 85(36), 82(8), 80(19), 78(9), 75(6), 70(17), 64(12), 60(16), 58(19), 56(3), 55(6), 50(17), 45(9), 40(6).

The data obtained above are summarized in Table 2:

Table 2: Questionnaire's data

Customers' Categories	L Class	R Class
A	60	60
B	40	90
C	20	45
D	30	45
E	20	15
Total	170	255

The evaluation of the above data (*verification* of the industry's decision about its new model) will be performed below in two ways:

I) Traditional (statistical) methods

a) Calculation of the means: It is straightforward to calculate the means m_L and m_R of the scores of the customers' answers for the Luxury and the Regular Class respectively, which are $m_L \approx 76.006$ and $m_R \approx 75.09$. This means that the customers were (marginally) very satisfied from their new cars, with the customers who purchased the L Class being a little bit better satisfied than those who purchased the R Class. Therefore the industry's decision for the circulation of its new model can be characterized as rather successful in general, although some improvements should be attempted in future for the cars of this new model and especially for the R Class.

b) Application of the GPA method: The *Great Point Average* (*GPA*) is a weighted mean where more importance is given to the higher scores achieved by the individuals of a, to which {high scores} greater coefficients (weights) are attached. In other words the GPA method, which is very frequently used in the USA, focuses to the quality "performance" rather, than to the mean "performance" of a group.

For applying the GPA method to the data of our example let us denote by n_A, n_B, n_C, n_D and n_E the number of the industry's customers who belong to the above described categories

A, B, C, D and E respectively and by n the total number of its customers. Then the GPA is calculated by the formula GPA = $\frac{n_D + 2n_C + 3n_B + 4n_A}{n}$. Obviously we have that 0 $\leq$ GPA $\leq$ 4.

In our case, using the data of Table 1 it is easy to check that both the GPA's of the customers of the L Class and of the R Class are equal to $\frac{43}{17} \approx 2.529$. This is a satisfactory value for the GPA, since it is close enough to 4. Thus, according to the GPA method the industry's customers of the L Class and of the R Class are equally satisfied with their new cars.

II) Application of the COG method (FL approach)

We consider as universal set the set U = {A, B, C, D, E} of the customers' categories. We are going to represent the sets L and R of the customers who purchased the L Class and R Class respectively as fuzzy sets in U. For this, we define the membership function $m: U \to [0, 1]$ for both sets L and R in terms of the frequencies, i.e. by $y = m(x) = \frac{n_x}{n}$, where n_x denotes the number of customers belonging to the category x in U and n denotes the total number of the customers of the corresponding set.

Then, from Table 2 it turns easily out that L and R can be written as fuzzy sets in U in the form[‡]: R = {(A, $\frac{6}{17}$), (B, $\frac{4}{17}$), (C, $\frac{2}{17}$), (D, $\frac{3}{17}$), (E, $\frac{2}{17}$)} (5)

and

L = {(A, $\frac{4}{17}$), (B, $\frac{6}{17}$), (C, $\frac{3}{17}$), (D, $\frac{3}{17}$), (E, $\frac{1}{17}$)} (6)

respectively

Next, we correspond to each $x \in U$ an interval of values from a prefixed numerical distribution as follows: E $\to$ [0, 1), D $\to$ [1, 2), C $\to$ [2, 3), B $\to$ [3,4), A $\to$ [4, 5]. This actually means that we replace U with a set of real intervals. Consequently, we have that y_1 = $m(x)$ = m(E) for all x in [0,1), y_2 = $m(x)$ = m(D} for all x in [1,2), y_3 = m(x) = m(C) for all x in [2, 3), y_4 = m(x) = m(B) for all x in [3, 4) and y_5 = m(x) = m(A) for all x in [4,5]. Since the membership values of the elements of U *in* L and R have been defined in terms of the corresponding frequencies, we obviously have that

$$\sum_{i=1}^{5} y_i = \text{m(A)} + \text{m (B)} + \text{m(C)} + \text{m(D)} + \text{m(E)} = 1 \qquad (7)$$

We are now in position to construct the graph of the membership function y = m(x), which has the form of the bar graph shown in Figure 2. From Figure 2 one can easily observe that the level's area, say F, contained between the bar graph of y = m(x) and the OX axis is

‡ We recall that a fuzzy set can be symbolically written in several forms, e.g. as a symbolic sum (see section 3), as a set of ordered pairs (see above), etc.

equal to the sum of the areas of the five rectangles F_i , i =1, 2, 3, 4, 5. The one side of each one of these rectangles has length 1 unit and lies on the OX axis.

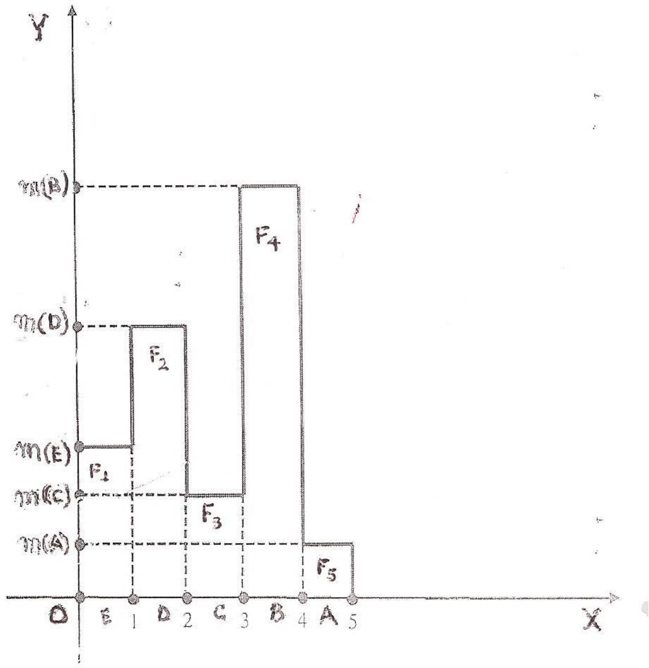

Figure 2: Bar graphical data representation

As it is well known from Mechanics, the coordinates (x_c, y_c) of the COG, say F_c, of the level's section F can be calculated by the formulas:

$$x_c = \frac{\iint\limits_F x dxdy}{\iint\limits_F dxdy}, y_c = \frac{\iint\limits_F y dxdy}{\iint\limits_F dxdy} \tag{8}$$

Taking into account the data of Figure 2 and equation (7) it is straightforward to check (see, for example, section 3 of [15]) that formulas (8) in our case can be transformed to the form:

$$x_c = \frac{1}{2}\left(y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5\right),$$
$$y_c = \frac{1}{2}\left({y_1}^2 + {y_2}^2 + {y_3}^2 + {y_4}^2 + {y_5}^2\right) \tag{9}$$

Then, using elementary algebraic inequalities it is easy to check that there is a unique minimum for y_c corresponding to COG $F_m(\frac{5}{2}, \frac{1}{10})$ ([15], section3). Further, the ideal case is when $y_1=y_2=y_3=y_4=0$ and $y_5=1$. Then from formulas (9) we get that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$. Therefore the COG in this case is the point $F_i(\frac{9}{2}, \frac{1}{2})$. On the other hand the worst case is when $y_1=1$ and $y_2=y_3=y_4= y_5=0$. Then from formulas (9) we find that the COG is the point $F_w(\frac{1}{2}, \frac{1}{2})$. Therefore the COG F_c of the level's section F lies in the area of the triangle $F_w F_m F_i$.

Then by elementary geometric observations one can obtain the following criterion ([15], section 3):

- *Among the two groups of the industry's customers the group with the biggest x_c corresponds to the customers who are better satisfied from their new cars.*
- *If the two groups have the same $x_c \geq 2.5$, then the group with the higher y_c corresponds to the customers who are better satisfied.*
- *If the two groups have the same $x_c < 2.5$, then the group with the lower y_c corresponds to the customers who are better satisfied.*

Substituting in formulas (9) the values of y_i's taken from the forms (5) and (6) of the fuzzy sets L and R respectively it is straightforward to check that the coordinate x_c of the COG for both L and R is equal to $\frac{103}{34} \approx 3.029 > 2.5$. However, the coordinate y_c is equal to $\frac{69}{578}$ for L and to $\frac{71}{578}$ for Rc. Therefore according to our criterion stated above, and in contrast to the conclusion obtained by calculating the corresponding means, the customers who purchased the R Class were better satisfied from their new cars.

5. Conclusions and Discussion

The following conclusions can be drawn from the investigation performed in this paper:

- We developed a Markov chain model for a mathematical description of the DM process. This model is an application of the Probability theory, which is based on principles of the classical bivalent logic.
- However, frequently in our everyday life a DM problem is expressed in an ambiguous way involving a degree of uncertainty. The process of DM under fuzzy conditions was also fully described in this paper through two representative examples.
- The verification of a taken decision is a step of the DM process, which is usually examined separately from its other steps, because it belongs to areas which, due to their depth and importance for the administrative rationalism, have become autonomous. In this paper we examined the verification of a taken decision both by traditional (statistical) methods (calculation of the mean – GPA) and by applying the COG deffuzification technique.

- The application of the above - three in total - methods for verifying a decision resulted to different conclusions in all cases! However, this is not embarrassing at all, since, in contrast to the calculation of the mean which focuses to the mean "performance" of a given group of individuals, the GPA and the COG methods focus on its quality "performance" by assigning weight coefficients to the higher scores obtained by these individuals. Further the COG method is more "sensitive" for the higher scores than the GPA does, since it assigns higher weight coefficients to them. Thus, it is suggested to the user of the above methods to choose the one that fits better to its personal criteria of goals.

We shall close with a brief discussion on our plans for future research on the subjects covered in this paper. First, there is a need to apply the methods developed in this paper on more real DM problems in order to get safer statistical results with respect to their applicability and their creditability in real situations.

Concerning our Markov model for the DM process, we must notice that we have applied similar models in the past to describe several situations in the areas of Management, Education and Artificial Intelligence (e.g. see book [18] and the relevant references appearing in it). Therefore it looks interesting and useful to search in future for more real situations where one could apply similar Markov models for their mathematical description and evaluation.

Finally, the special form of the COG defuzzification technique that we have used in this paper for examining the verification of a taken decision it turns out to be *a general assessment method* [13], that could be utilized in many other real situations characterized by a degree of ambiguity and/or uncertainty, apart from those where we have already applied it ([10-17], etc).

References

[1] Berger J. O., "Statistical Decision Theory: Foundations, Concepts and Methods", Springer- Verlag, New York, 1980.

[2] Zadeh L., A., "Fuzzy Sets", Information and Control, 8, 338-353,1965

[3] Taha, H. A., "Operations Research – An Introduction", Second Edition, Collier Macmillan, New York - London, 1967.

[4] Kemeny, J. G., Schleifer A. Jr. & Snell, J. L, "Finite Mathematics with Business Applications", Prentice Hall, London, 1962.

[5] Kemeny J. G. & Snell J. L., "Finite Markov Chains", New York, Springer - Verlag, 1976,

[6] Voskoglou M. Gr. & Perdikaris S. C., "A Markov Chain model in Problem-Solving", International .Journal of Mathematics Education in Science and Technology, 22, 909-914, 1991

[7] Klir, G. J. & Folger, T. A., “Fuzzy Sets, Uncertainty and Information”, Prentice-Hall, London, 1988.

[8] Bellman, R. E. & Zadeh, L. A., “Decision making in fuzzy environment”, Management Science, 17, 141-164, 1970 .

[9] van Broekhoven, E. & De Baets, B., “Fast and accurate centre of gravity defuzzification of fuzzy system outputs defined on trapezoidal fuzzy partitions”, Fuzzy Sets and Systems, 157(7), 904-918, 2006

[10] Subbotin, I., Badkoobehi, H. & Bilotskii, N., “Application of Fuzzy Logic to Learning Assessment”, Didactics of Mathematics: Problems and Investigations, 22, 38-41, 2004.

[11] Subbotin, I., Mossovar-Rahmani, F. & Bilotskii, N., “Fuzzy logic and the concept of the Zone of Proximate Development”, Didactics of Mathematics: Problems and Investigations, Volume 36, 101- 108, 2011.

[12] Subbotin, I. Ya, & Voskoglou, M. Gr., “Applications of Fuzzy Logic to Case-Based Reasoning”, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 1, 7-18, 2011.

[13] Voskoglou, M. Gr., “A study on Fuzzy Systems”, American Journal of Computational and Applied Mathematics, 2(5), 232-240, 2012.

[14] Voskoglou, M. Gr. & Subbotin, I. Ya., “Fuzzy Models for Analogical Reasoning”, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 2, 19-38, 2012.

[15] Voskoglou, M. Gr., “Problem Solving, Fuzzy Logic and Computational Thinking”, Egyptian Computer Science Journal, 37(1), 131-145, 2013.

[16] Subbotin, I. Ya, & Voskoglou, M. Gr., “Language, mathematics and critical thinking: The cross influence and cross enrichment”, Didactics of Mathematics: Problems and Investigations, 41, 89- 94, 2014.

[17] Voskoglou, M. Gr., “Assessing the Players’ Performance in the Game of Bridge: A Fuzzy Logic Approach“, American Journal of Applied Mathematics and Statistics, 2(3), 115-120, 2014.

[18] Voskoglou, M. Gr., “Stochastic and fuzzy models in Mathematics Education, Artificial Intelligence and Management”, Lambert Academic Publishing, Saarbrucken, Germany, 2011 (for more details look at http://amzn.com./3846528218).