A Tool for Assessing the Ability of Understanding the Infinity Based on the Triangular Fuzzy Numbers

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Abstract

From the instructor's optical corner the student understanding of the infinite is characterized by a degree of vagueness. Therefore fuzzy logic, due to its property of assigning multiple values to the ambiguous cases, could help for a more effective study of the student difficulties to deal with the infinite. Under this sceptic, we utilize here the triangular fuzzy numbers as an assessment tool in an experimental study on the effects that an instruction to the basic philosophical / epistemological aspects about the infinite could have for the improvement of student abilities to deal successfully with the concept of infinity in their mathematical courses.

Keywords: *Potential/Actual Infinity, Unattainable Infinity, Student Assessment, Fuzzy Assessment Methods, Triangular Fuzzy Numbers (TFNs).*

1. Introduction

Philosophers, mathematicians, mathematical historians and educators, students and many others have struggled for centuries to resolve the various issues and paradoxes regarding conceptions of the infinity. Aristotle's (384-322 BC) *potential/actual dichotomy* dominated these conceptions for centuries. According to his view, the potential infinity could be understood as the infinite presented over time, while the actual infinity is the infinite present at a moment in time. For Aristotle the actual infinity is incomprehensible, because the underlying process of such an actuality would require the whole of time. This distinction of the concept of infinity allowed Aristotle to acknowledge the existence of the infinite, provided that it was not present "all at once" ([1], p. 39). Further, the actual infinity explains, according to him, all the paradoxes connected to the infinite.

However, views also appeared disputing the ideas of Aristotle, mainly expressed by the *rationalists*, who believed that we can invoke the pure logic for the understanding of the real world in general and the actual infinity in particular. Bolzano (1741-1848) advanced, against the empiricist Aristotle's negative assertion, the idea of the existence of an *infinite collection* as a completed whole. His main argument to support this view was the existence of the *large finite numbers*, like the grains of sand in a desert, a set with $10^{10^{10^{10^{10}}}}$ elements, etc, which, although they doubtlessly exist, they cannot been enumerated by human beings. However, one concern with Bolzano's view is that the examples he used are finite sets. For instance, in case of enumerating the set of the first $10^{10^{10^{10^{10}}}}$ natural numbers one can reflect on the last counting number as indicating its cardinality, a fact which cannot occur in an infinite set, where there is no such number.

Cantor (1845-1918) extended Bolzano's thinking. His theory of *transfinite numbers* is connected to his view that infinite sets to which a cardinality or order can be assigned "enjoy a kind of finitude" or are "really finite". Cantor thus suggests three cognitive categories, the *finite*, the *attainably infinite* and the *unattainably infinite*. The last one, termed by Moore [1] as the "really infinite", refers to immeasurably large collections to which no cardinality or order can be assigned, like the collection of everything thinkable, the set of all the sets, etc. According to Cantor, actual infinite entities are considered to be attainably infinite, while potentially infinite collections that cannot be actualized are considered to be unattainably infinite.

Nowadays, the best way for connecting the potential to the actual infinity is probably the use of *fractals* [2], which are obtained by infinite processes characterized by a kind of self – similarity. Consider, for example, the *ternary set* discovered by Henry John Stephen Smith in 1874, but better known as *the Cantor's comb or dust*. This set, through the consideration of which Cantor (1883) and others helped for laying the foundations of the modern point-set Topology, is created by removing repeatedly the open middle thirds of a line segment [3]. The first five steps of this construction are represented in Figure 1.

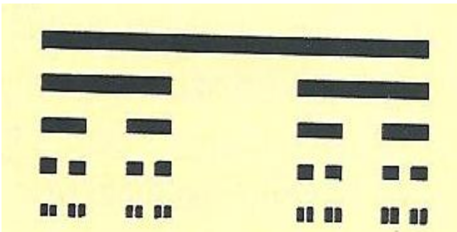

Figure 1: Graph of the ternary set

Figure 1 does not represent the set's final image, the creation of which requires an infinite number of such steps (actual infinity); it gives however a very precise approximation of it. In fact, it is easy to observe that the left and right parts of Figure 1 are similar, containing equal lengths. Further, each of these parts is similar to the whole figure and it also contains its own left and right parts. Therefore we have 4, 8, 16, ,,,,,, etc, smaller subsets similar to the original set. As the process continues, it becomes evident that the ternary set contains an infinite number of smaller and smaller subsets, all of which are similar to the original set (self-similarity). Cantor's comb is probably the first fractal discovered in the history of mathematics.

However, although nowadays in just about every case there is a rigorous mathematical explanation, many students have considerable difficulty in understanding the infinite. Tsamir [4], for example, found that prospective teachers erroneously attribute properties of finite to infinite sets, Mamona-Downs [5] found that many students consider that the limit of a sequence is its last term and, given the sequence (a_n), $n \in \boldsymbol{N}$, they write a_∞ for its limit, etc.

Dubinsky et al. [6] analyzed the difficulties appearing to individuals for understanding the concept of infinity in terms of their *APOS theory* for teaching/learning mathematics, developed during the 1990's in the USA (eg. see [7-10], etc). According to this theory, an individual deals with a mathematical situation by using the mental mechanisms of *interiorization* and *encapsulation* to build cognitive structures that applied to the situation. The related structures involve *actions*, *processes*, *objects* and *schemas* and the word APOS is an acronym formed by the initial letters of these words. According to the APOS theory [6], one's ability to perform isolated steps of an infinite process is an action, while the interiorization of this action to a process implies the individual's ability of repeating mentally this action for an unlimited number of steps (potential infinity). Further, the actual infinity involves the understanding of an infinite process as a totality (Bolzano) and the encapsulation of this totality to a cognitive object (Cantor), i.e. the actual infinity is an attainable form of the infinite. However, the understanding of a process as a totality and therefore its encapsulation to an object is not always possible, which means that the unattainable infinite is a form of potential infinity that cannot be understood as a totality. Conclusively the potential and actual infinity are two different cognitive conceptions of the infinite, which, in an advanced phase of the individual's cognitive progress, are embodied together in his/her corresponding cognitive schema. Obviously the existence of the one does not deny the existence of the other, neither is a wrong conception of the other. The relationship between them can be better understood through the transformation from an infinite process (e.g. a sequence) to the final result obtained by the encapsulation of this process to an object (e.g. limit of the sequence). This result *transcends* in general the corresponding process, in the sense that it is not connected, neither is obtained by any of its steps. This is the characteristic difference between the large finite numbers and the infinite, which explains why the former can be more easily understood than the latter one.

Fuzzy logic, due to its property of characterizing the uncertain situations with multiple values offers rich resources for the evaluation of such kind of situations. Consequently, since from the instructor's optical corner the understanding of the infinity by students' is characterized by a degree of vagueness, the application of *fuzzy assessment methods* (e.g. [11-14], etc) could help for a more effective study of student skills to deal successfully in their mathematical courses with situations in which the infinite is involved.

In this work we utilize the *Triangular Fuzzy Numbers* (TFNs) as a tool for assessing the degree of student understanding of the infinite. The rest of the paper is formulated as follows: In Section 2 we present the basics from the TFNs needed for our purposes. In Section 3 we describe a classroom experiment performed with first year university students and we use the TFNs for the assessment of their skills to deal successfully with the infinite. The creditability of our fuzzy model is checked through the parallel use of two other, traditional assessment methods, i.e. the *calculation of the mean values* of the student grades and the *Grade Point Average* (GPA) index. Finally, Section 4 is devoted to our conclusions and a brief discussion on the perspectives of future research on the subject.

2. Triangular Fuzzy Numbers

2.1 Fuzzy Numbers (FNs)

FNs play an important role in fuzzy mathematics, analogous to the role played by the ordinary numbers in classical mathematics. The definition of a FN is the following:

2.1.1 Definition: A FN is a *fuzzy set* A on the set ***R*** of real numbers with membership function m_A: ***R*** $\rightarrow$ [0, 1], such that:

- A is *normal*, i.e. there exists x in ***R*** such that $m_A(x) = 1$,
- A is *convex*, i.e. all its a-cuts $A^a = \{x \in U: m_A(x) \geq a\}$, a in [0, 1], are closed real intervals, and
- Its membership function $y = m_A(x)$ is a *piecewise continuous* function.

Figure 2 represents the graph of a fuzzy set on ***R*** which is not convex. For example, we observe that $A^{0.4} = [5, 8.5] \cup [11, 13]$, i.e. $A^{0.4}$ is not a closed interval.

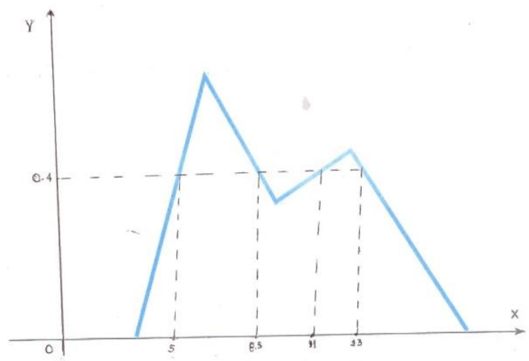

Figure 2: Example of a non convex fuzzy set on *R*

Since the x-cuts A^x of a FN, say A, are closed real intervals, we can write $A^x = [A_l^x, A_r^x]$ for each x in [0, 1], where A_l^x, A_r^x are real numbers depending on x.

The following statement defines a *partial order* on the set of all FNs:

2.1.2 Definition: Given the FNs A and B we write $A \leq B$ (or $\geq$) if, and only if, $A_l^x \leq B_l^x$ and $A_r^x \leq B_r^x$ (or $\geq$) for all x in [0, 1]. Two such FNs are called *comparable*, otherwise they are called *non comparable*.

2.1.3 Remark: One can define the four basic *arithmetic operations* on FNS in the following two, equivalent to each other, ways [15]:

(i) With the help of their a-cuts and the Representation-Decomposition Theorem of Ralesscou-Negoita ([16], Theorem 2.1, p.16) for fuzzy sets. In this way the fuzzy arithmetic is turned to the well known arithmetic of the closed real intervals.

(ii) By applying the Zadeh's extension principle ([17], Section 1.4, p.20), which provides the means for any function f mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.

In practice the above two general methods of the fuzzy arithmetic, requiring laborious calculations, are rarely used in applications, where the utilization of simpler forms of FNs is preferred.

For general facts on FNs we refer to Chapter 3 of the book of Theodorou [18], which is written in Greek language, and also to the classical on the subject book of Kaufmann and Gupta [15].

2.2 Triangular Fuzzy Numbers (TFNs)

TFNs are the simplest form of FNs. A TFN (*a, b, c*), with *a, b, c* in ***R*** actually means that "the value of *b* lies in the interval [*a, c*]". The membership function of (*a, b, c*) is zero outside the interval [*a, c*], while its graph in [*a, c*] consists of two straight line segments forming a triangle with the OX axis (Figure 3). Therefore the analytical definition of a TFN is given as follows:

2.2.1 Definition: Let *a, b* and *c* be real numbers with *a* < *b* < *c*. Then the *TFN* (*a, b, c*) is a FN with membership function:

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

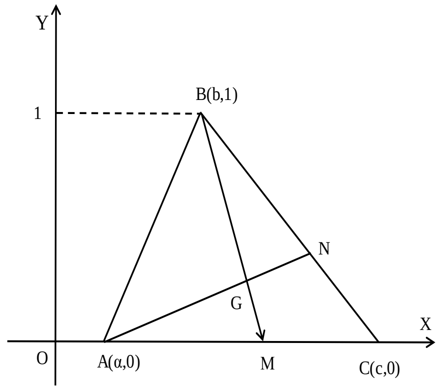

Figure 3: Graph and COG of the TFN (a, b, c)

The following two Propositions refer to basic properties of TFNs that we are going to use later in this paper:

2.2.2 Proposition: The x-cuts A^x of a TFN A = (a, b, c), x$\in$[0, 1], are calculated by the formula $A^x = [A_l^x, A_r^x] = [a + x(b - a), c - x(c - b)]$.

Proof: Since $A^x = \{y \in \boldsymbol{R}: m(y) \geq x\}$, Definition 2.2.1 gives for the case of A_l^x that $\frac{y-a}{b-a} = x \Leftrightarrow y = a + x(b - a)$. Similarly for the case of A_r^x we have that $\frac{c-y}{c-b} = x$ $\Leftrightarrow y = c - x(c - b)$.

2.2.3 Proposition: *(Defuzzification of a TFN)* The coordinates (X, Y) of the COG of the graph of the TFN (a, b, c) are calculated by the formulas $X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$.

Proof: The graph of the TFN (a, b, c) is the triangle ABC of Figure 3, with A (a, 0), B(b, 1) and C (c, 0). Then, the COG, say G, of ABC is the intersection point of its medians AN and BM. The proof of the Proposition is easily obtained by calculating the equations of AN and BM and by solving the linear system of these two equations.

2.2.4 Arithmetic Operations on TFNs: It can be shown [15] that the two general methods of defining arithmetic operations on FNs mentioned in Remark 2.1.3 lead to the following simple rules for the *addition* and *subtraction* of TFNs:

Let A = (a, b, c) and B = (a_1, b_1, c_1) be two TFNs. Then

- The sum A + B = ($a+a_1$, $b+b_1$, $c+c_1$).
- The difference A - B = A + (-B) = ($a-c_1$, $b-b_1$, $c-a_1$), where –B = ($-c_1$, $-b_1$, $-a_1$) is defined to be the *opposite* of B.

In other words, the opposite of a TFN, as well as the sum and the difference of two TFNs are always TFNs. On the contrary, the *product* and the *quotient* of two TFNs, although they are FNs, they are not always TFNs, unless if a, b, c, a_1, b_1, c_1 are in $\boldsymbol{R}^+$ ([14], Section IV).

One can also define the following two *scalar operations*:

- k + A= (k+a, k+b, k+c), k$\in \boldsymbol{R}$
- kA = (ka, kb, kc), if k>0 and kA = (kc, kb, ka), if k<0.

We close this section with the following definition, which is introduced to be used in Section 3 for assessing the student understanding of the infinite with the help of TFNs:

2.2.5 Definition: Let A_i , i = 1, 2,…, n be TFNs, where n is a non negative integer, $n \geq 2$. Then we define the *mean value* of the A_i's to be the TFN:

$$A = \frac{1}{n}(A_1 + A_2 + \ldots + A_n).$$

3. The Classroom Experiment

One can find in the literature reflections of the development of the concept of infinity in students of today ([4, 19, 20], etc). Doubtlessly, the pioneer of this study was *E. Fischbein*, whose empirical researches revealed many conflicting intuitional student perceptions of the infinite [21-25]. His last article [25] was published just after his death, in 2001, together with six articles of other authors [4, 5, 26-29] in a special issue of the "Educational Studies of Mathematics" on the concept of infinity, dedicated to his memory.

The impulsion to perform the following classroom experiment was given by our concern to study the effects that an instructor's lecture to students on the basic philosophical/epistemological aspects of the infinite could have for the improvement of their abilities to deal successfully in their mathematical courses with situations involving the concept of infinity. For this, we selected two equivalent - according to the marks obtained in their first term course "Higher Mathematics I"- student groups from the School of Technological Applications (prospective engineers) of the Graduate Technological Educational Institute (T. E. I.) of Western Greece (in the city of Patras) being at their second term of studies. A two hours lecture was delivered separately to the students of both groups. The lecture to the first (experimental) group was focused mainly on the basic philosophical/epistemological aspects of the infinite (see Section 1), while the attention of the lecture for the second (control) group was turned to examples related to the topics of the course "Higher Mathematics I" * involving, directly or indirectly, the concept of infinity. Next, a written test was performed for both groups in terms of the questionnaire presented in the Appendix at the end of the paper together with some representative wrong answers. The student answers were marked in a climax from 0 to 100 and the scores obtained are the following:

Group 1 (G_1): 100(5 times), 99(3), 98(10), 95(15), 94(12), 93(1), 92 (8), 90(6), 89(3), 88(7), 85(13), 82(4), 80(6), 79(1), 78(1), 76(2), 75(3), 74(3), 73(1), 72(5), 70(4), 68(2), 63(2), 60(3), 59(5), 58(1), 57(2), 56(3), 55(4), 54(2), 53(1), 52(2), 51(2), 50(8), 48(7), 45(8), 42(1), 40(3), 35(1).

Group 2 (G_2): 100(7), 99(2), 98(3), 97(9), 95(18), 92(11), 91(4), 90(6), 88(12), 85(36), 82(8), 80(19), 78(9), 75(6), 70(17), 64(12), 60(16), 58(19), 56(3), 55(6), 50(17), 45(9), 40(6).

The following linguistic characterizations (grades) were assigned to the above scores: A (100-85) = excellent, B (84-75) = very good, C (60-74) = good, D(50-59) = fair and F (<50) = not satisfactory. The student results with respect to the above grades are depicted in Table 1.

* The course involves an introductory chapter on the basic sets of numbers, Differential and Integral Calculus in one variable and elements of Analytic Geometry and Linear Algebra.

Table 1: Characterization of the student performance

Grade	G_1	G_2
A	60	60
B	40	90
C	20	45
D	30	45
E	20	15
Total	170	255

The overall performance of the two student groups was evaluated first by two traditional assessment methods and finally by using the TFNs as assessment tools:

i) Mean values: A straightforward calculation gives that the mean values of the above student scores are approximately equal to 76.006 and 75.09 for G_1 and G_2 respectively. This shows that the *mean performance* of both student groups can be characterized (on the boundary) as very good, with the performance of the experimental group G_1 being slightly better.

ii) GPA index: We recall that the *Grade Point Average (GPA)* index is a weighted mean, in which more importance is given to the higher scores, by assigning greater coefficients (weights) to them. In other words, the GPA index measures the *quality performance* of a student group. For calculating the GPA index let us denote by n_A, n_B, n_C, n_D and n_F the numbers of students whose performance was characterized by A, B, C, D and F respectively and by n the total number of students of each group. .It is well known then that the GPA index is calculated by the formula GPA= $\frac{0n_F + n_D + 2n_C + 3n_B + 4n_A}{n}$ (1); e.g. see [30].

Formula (1) gives that, GPA=0, if $n_F = n$ (worst case) and GPA=4, if $n_A = n$ (ideal case). Therefore $0 \leq$ GPA ≤ 4, which implies that values of GPA greater than the half of its maximal value, i.e. greater than 2, could be considered as being connected to a satisfactory group's performance.

In our case, applying formula (1) on the data of Table 1, one finds that the GPA index for both groups is equal to $\frac{43}{17} \approx 2.529$. Thus, the two student groups demonstrated the same, satisfactory, quality performance.

(iii) Use of the TFNs as Assessment Tool: We assign to each linguistic label (grade) a TFN (denoted by the same letter) as follows: A= (85, 92.5, 100), B = (75, 79.5, 84), C = (60, 67, 74), D= (50, 54.5, 59) and F = (0, 24.5, 49)[†]. The middle entry of each of the above TFNs is equal to the mean value of the student scores that we have previously attached to

[†] The representation of the linguistic labels A, B, C, D and F by TFNs has the advantage of determining numerically the scores corresponding to each grade. In fact, the scores assigned to the above grades in our example are not standard, since they may differ from case to case. For example, in a more rigorous assessment one could take A(90-100), B (80-89), C(70-79), D (60-69), F(<60), etc.

the corresponding grade. In this way a TFN corresponds to each student assessing his (her) individual performance

We observe now that in Table 1 we actually have 170 TFNs representing the progress of the students of G_1 and 255 TFNs representing the progress of the students of G_2. Therefore, it is logical to accept that the overall performance of each student group is given by the corresponding mean value of the above TFNs (Definition 2.2.5). For simplifying our notation, let us denote the above mean values by the letter of the corresponding student group. Then, making straightforward calculations, we find that

$$G_1 = \frac{1}{170}.(60A+40B+20C+30D+20F) \approx (63.53, 71.74, 83.47) \text{ and}$$

$$G_2 = \frac{1}{255}.(60A+90B+45C+45D+15F) \approx (65.88, 72.63, 79.53).$$

Observing the left entries (63.53 and 65.88 respectively) and the right entries (83.47 and 79.53 respectively) of the TFNs G_1 and G_2 one concludes that the overall performance of the two student groups could be characterized from good (C) to very good (B).

It is also of worth to clarify that the middle entries of G_1 and G_2 (71.74 and 72.63 respectively) give *a rough approximation* only of each Department's overall performance. In fact, since the middle entries of the TFNs A, B, C, D and F were chosen to be equal to the means of the scores assigned to the corresponding linguistic grades, the middle entries of the TFNS G_1 and G_2 are simply equal to the mean values of these means and therefore they *do not measure the mean performances* of the two student groups.

Next, applying Proposition 2.2.2 one finds that the x-cuts of the two TFNs are G_1^x = [63.53+8.21x, 83.47-11.73x] and G_2^x = [65.88+6.75x, 79.53-6.9x] respectively. But $63.53+8.21x \leq 65.88+6.75x \Leftrightarrow 1.46x \leq 2.35 \Leftrightarrow x \leq 1.61$, which is true, since x is in [0, 1]. On the contrary, $83.47-11.73x \leq 79.53-6.9x \Leftrightarrow 3.94 \leq 4.83x \Leftrightarrow 0.82 \leq x$, which is not true for all the values of x. Therefore, according to Definition 2.1.2 the TFNs G_1 and G_2 are not comparable, which means that at this stage one *can not decide which of the two groups demonstrates the better performance*.

A good way to overcome this difficulty is to *defuzzify* the TFNs G_1 and G_2. By Proposition 2.2.3, the COGs of the triangles forming the graphs of the TFNs G_1 and G_2 have x-coordinates equal to $X = \frac{63.53+71.74+83.47}{3} \approx 72.91$ and $X' = \frac{65.88+72.63+79.53}{3} \approx 72.68$ respectively.

Observe now that the GOGs of the graphs of G_1 and G_2 lie in a rectangle with sides of length 100 units on the X-axis (student scores from 0 to 100) and one unit on the Y-axis (normal fuzzy sets). Therefore, *the nearer the x-coordinate of the COG to 100, the better the corresponding group's performance*, Thus, since $X > X'$, G_1 demonstrates a (slightly) better overall performance than G_2.

Normally, the performance of the control group was expected to be better than that of the experimental group, since its students were exposed during the two hours extra lecture to examples connected to the infinite. Thus, the fact that the experimental group demonstrated a better mean performance and the same quality performance with the control group, means that, at least its mediocre students, benefited by the instructor's presentation of the basic philosophical/epistemological aspects of the infinite. However, the conclusions of the above experiment are not statistically safe, because the differences found in the performances of the two groups were small enough.

vi) Remark: An alternative approach for ranking the TFNs G_1 and G_2 of paragraph (iii) could be the use of the *Yager index* introduced for ordering fuzzy subsets of the unit interval [31].

4. Discussion and Conclusions

The following conclusions can be drawn from the material presented in this paper:

- Students of today are facing significant difficulties for understanding the concept of infinity and especially the actual form of it, according to the Aristotle's dichotomy.

- A classroom experiment performed recently by the author and presented in this work has shown that an instruction to the basic philosophical/epistemological aspects of infinity could benefit students to deal successfully in their mathematical courses with cases involving, directly or indirectly, the concept of infinity.

- Fuzzy logic, due to its property of characterizing the ambiguous situations with multiple values, offers rich resources for the evaluation of such kind of situations. Consequently, since from the instructor's optical corner the understanding of the infinity by students' is characterized by a degree of vagueness, the idea of utilizing the TFNs as an assessment tool in our classroom experiment had a strong logical base. The application of this idea was proved to be successful, validated by two traditional assessment methods: the calculation of the mean values and the GPA index

Two are the main objectives of our future research on the subject. First, since the differences in the performance of the two groups found in our experiment were small enough, the conclusions obtained are not statistically safe and therefore *further experimental research* is needed. On the other hand, the use of TFNs as an assessment tool seems to have the potential of a *general assessment method* that could be used in future for the assessment of various other human activities and for the evaluation of the effectiveness of several intelligent systems for Case-Based Reasoning, Decision Making, etc. The *Trapezoidal FNs* (TpFNs) ([14], Section IV) being a generalization of TFNs, as well as other forms of FNs could be also used for assessment purposes.

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Appendix: Questionnaire of the Experiment and Representative Wrong Answers

I. Questionnaire

1. a) Compare the numbers 4.9999….. and 5.

   b) Are there any fractions between $\frac{1}{10}$ and $\frac{1}{11}$? If yes, write one of them.

2. Compare the cardinalities of the sets N of natural numbers, N_E of the even natural numbers, Z of the integers, Q of the rational and R of the real numbers. Justify your answers.

3. Examine if there exist the limits: a) $\lim_{x\to 2}\sqrt{x^2-9}$, b) $\lim_{x\to a} f(x)$, with $f(x)=\begin{cases}1, & x\in Q\\ 0, & x\in R-Q\end{cases}$, $a\in R$, where Q is the set of rational and R is the set of real numbers.

4. Given the line segment AB with length 1 m we add to it the line segments BC of length $\frac{1}{2}$ m, CD of length $\frac{1}{4}$ m, DE of length $\frac{1}{8}$ m, EG of length $\frac{1}{16}$ m,…. and so on. Find the total length of AB + BC + CD + DE + EG +….. (This problem was retrieved from [4]).

5. Starting from the interval [0, 1] we delete first its middle third ($\frac{1}{3}$, $\frac{2}{3}$), then the middle thirds ($\frac{1}{9},\frac{2}{9}$) and ($\frac{7}{9},\frac{8}{9}$) of the two remaining intervals [0, $\frac{1}{3}$] and [$\frac{2}{3}$, 1] respectively, and so on (Cantor's comb: See Section 1).

   a) Find the total length of the removed intervals when the above process is repeated infinitely many times (the lengths of the removed intervals form a geometric progression with first term equal to $\frac{1}{3}$ and ratio $\frac{2}{3}$, therefore their infinite sum is 1).

   b) Are there any points left behind in this case?

II. Wrong Answers

1. a) 5 is greater than 4.9999……

   b) No, because $\frac{1}{11}$ is the fraction next to $\frac{1}{10}$.

2. Since $N_E \subset N$, N has a greater cardinality, etc. Also: All these sets are infinite and therefore they have the same cardinality, which is equal to ∞, or they have no cardinality, which, in case of existence, should be a real number.

3. a) The limit does not exist, because $2^2 - 9 < 0$ and the negative numbers have not real square roots.

   b) There are two limits equal to 0 and 1 respectively.

4. The total length is infinite, since the successive additions are repeated infinitely many times.

5. a) The total length removed is less than 1, because there are some points of the initial interval [0, 1] left behind, like $\frac{1}{3}$, $\frac{2}{3}$, etc.

   b) There are no points left behind, since the total length removed is equal to 1.