Determination of Fundamental Units of Real Quadratic Number Fields Related with Specific Continued Fraction Expansions

Özen ÖZER

Department of Mathematics, Faculty of Science and Arts, Kırklareli University

Kırklareli, 39100, Turkey

ozenozer39@gmail.com

Abstract

Integral basis element of real quadratic field $Q(\sqrt{d})$ has two different types according to $d \equiv 2,3(mod4)$ or $d \equiv 1(mod4)$. The present paper deals with classifying the real quadratic number fields $Q(\sqrt{d})$ having continued fraction expansion of algebraic integer w_d with repeated 4s in symetric part of the period length$\ell(d)$ where $d \equiv 2,3(mod4)$ is a square free positive integer.

General explicit parametric representations are considered as theorems to determine fundamental unit ϵ_d of such real quadratic number fields $Q(\sqrt{d})$ along with parametrized forms of d which are still missing in the open literature. Moreover, Yokoi's d-invariants n_d and m_d in the relation to continued fraction expansion of w_d are given by using coefficient of fundamental unit of such real quadratic fields..

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1 Introduction

Let $k = Q(\sqrt{d})$ be a real quadratic number field where $d > 0$ is a positive square-free integer.Continued fraction expansion of integral basis element $w_d = \sqrt{d}$ of $Z[\sqrt{d}]$ is represented as $w_d = \left[a_0; \overline{a_1, a_2, \ldots, a_{\ell(d)-1}, 2a_0}\ \right]$ where period length is $\ell(d)$ for $d \equiv 2,3(mod4)$. The fundamental unit ϵ_d of real quadratic number field is also denoted by $\epsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$ where $N(\epsilon_d) = (-1)^{\ell(d)}$. Also, Yokoi's invariants are expressed by $n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right]$ and $m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right]$ where $[[x]]$ represents the floor of x.

Information about continued fraction expansion is in the excellent books [8], [11], [15] and [17].Sasaki [16] and Mollin [8-10] worked on lower bound of fundamental unit for real quadratic number fields as well as class numbr one or two and desired certain important results. In [20,] William and his colleggue compared the length of continued fraction of integral basis element according to $\sqrt{d}$ and $(1+\sqrt{d})/2$.Yokoi [21-24] defined several invariants important for class number problem and solutions of Pell equation by using coefficients of fundamental unit. Tomita and Yamamuro [19] got some significant results for fundamental unit by using the terms of Fibonacci sequence. Tomita also determined the fundamental unit and class number by using continued fraction expansion of integral basis element for period length equal to 3 in [18]. Lately,Tomita and his co-researcher Kawamato [5] proved a relation between real quadratic fields of class number one anda misterious behaviour of the simple continued fraction expansion of certain quadratic irrationals In [12-14], the author worked on some special types of real quadratic fields when integral basis element has even partial constant elements. For more information and background, the readers can look the [1-4] and [6-7] references with details.

In real quadratic fields $Q(\sqrt{d})$, d has several different forms in the terms of H_i sequences according to partial constant elements equal to each others and written by even integers or odd integers for $d \equiv 2, 3(mod4)$.

In this paper an attempt has been made to determine the continued fraction expansion which has partial quotient elements are equal each other and written as 4s(except the last digit of the period, which is always $2\left[\left[\sqrt{d}\right]\right]$ for $w_d = \sqrt{d}$ for $w_d = \sqrt{d}$) in the symetric part of period length.

Infinitely many values of d having 4s in the symetric part of period length are encountered.Here we classify them with regard to arbitrary period length. This will help to us determine the general form of fundamental units ϵ_d as well as the form of continued fraction expansions.

Also, the results obtained on fundamental units, period length and Yokoi's invariants n_d, m_d will be demonstrated with their several values in the tables as corollories.

2 Preliminaries

Definition 2.1. $\{H_i\}$ is a sequence defined by the recurrence relation

$$H_i = 4H_{i-1} + H_{i-2}$$

for $i \geq 2$ with seed values $H_0 = 0$ and $H_1 = 1$. Several values of this sequence can be calculated as follows:

$H_2 = 4$, $H_3 = 17$ $H_4 = 72$, $H_5 = 305$, $H_6 = 1292$, $H_7 = 5473$, $H_8 = 23184$, $H_9 = 98209$

$H_{10} = 416020$, $H_{11} = 1762289$, $H_{12} = 7465176$, $H_{13} = 31622993$, $H_{14} = 158114965, \ldots$

Parametrization of d is written by using the terms of this sequence in the next section.

Definition 2.2. Let $c_n = ac_{n-1} + bc_{n-2}$ recurrence relation of $\{c_n\}$ sequence where a, b are real numbers.The polynomial is called as a characteristic equation is written in the form:

$$x^2 - ax - b = 0$$

for $\{c_n\}$ sequence. By using the definition, we find characteristic equation as

$$x^2 - 4x - 1 = 0$$

for $\{H_k\}$ sequence. So, each element of sequence can be written as:

$$H_k = \frac{1}{2\sqrt{5}} \left[\left(2+\sqrt{5}\right)^k - \left(2-\sqrt{5}\right)^k\right]$$

for $k \geq 0$.

Remark 2.3. Let $\{H_n\}$ be the sequence defined as in Definition 2.1. Then, we state the following:

$$H_n \equiv \begin{cases} 0\,(mod4)\,, & n \equiv 0\,(mod2); \\ 1\,(mod4)\,, & n \equiv 1\,(mod2). \end{cases}$$

for $n \geq 0$.

Lemma 2.4. For a square-free positive integer d congruent to $2, 3$ modulo 4, we put $\omega_d = \sqrt{d}$, $a_0 = [\omega_d]$, $\omega_R = a_0 + \omega_d$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period $l = l(d)$ of ω_R, we get $\omega_R = [\overline{2a_0, a_1, ..., a_{l-1}}]$ and $\omega_d = [a_0, \overline{a_1, ..., a_{l-1}, 2a_0}]$. Furthermore, let $\omega_R = \frac{(P_l \omega_R + P_{l-1})}{(Q_l \omega_R + Q_{l-1})} = [2a_0, a_1, ..., a_{l-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ϵ_d of $Q\left(\sqrt{d}\right)$ is given by the following formula:

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{\ell(d)} + Q_{\ell(d)-1}\rangle 1$$

and

$$t_d = 2a_0.Q_{\ell(d)} + 2Q_{\ell(d)-1}, \quad u_d = 2Q_{\ell(d)}.$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_i Q_i + Q_{i-1}$, $(i \geq 1)$.

Proof. Proof is omitted in [19, Lemma 1]. □

Lemma 2.5. Let d be the square free positive integer congruent to $2, 3$ modulo 4 .We consider w_d which has got partial constant elements repeated 4s in the case of period $l = l(d)$. If a_0 denote the integer part of w_d for d congruent to $2, 3(mod4)$, then we have continued fraction expansions

$$w_d = \sqrt{d} = \left[a_0; \overline{a_1, a_2, \ldots, a_{\ell(d)-1}, a_{\ell(d)}}\ \right] = \left[\ a_0; \overline{4, 4, \ldots, 4,\ 2a_0}\ \right]$$

for quadratic irrational numbers and $w_R = a_0 + \sqrt{d} = \left[\overline{2a_0, 4, \ldots, 4}\ \right]$ for reduced quadratic irrational numbers.

Moreover, $A_j = a_0 H_{j+1} + H_j$ and $B_j = H_{j+1}$ are determined in the continued fraction expansions where $\{A_j\}$ and $\{B_j\}$ are two sequences defined by:

$$A_{-2} = 0, \quad A_{-1} = 1, \quad A_j = a_j.A_{j-1} + A_{j-2},$$

$$B_{-2} = 1, \quad B_{-1} = 0, \quad B_j = a_j.B_{j-1} + B_{j-2},$$

for $j \geq 0$ and $j < l(d)$ where $l(d)$ is period length of w_d. Then $C_j = A_j/B_j$ is the j^{th} convergent in the continued fraction expansion of $\sqrt{d}$. Moreover, $A_l = 2a_0^2 H_l + 3a_0 H_{l-1} + H_{l-2}$ and $B_l = 2a_0 H_l + H_{l-1}$ for $j = l(d)$

Furthermore,in the continued fraction $w_R = a_0 + \sqrt{d} = [b_1, b_2, \ldots, b_n, \ldots] = [2a_0, 4, \ldots, 4, \ldots]$, $P_k = 2a_0 H_k + H_{k-1}$ and $Q_k = H_k$ are determined in the continued fraction expansion where $\{P_k\}$ and $\{Q_k\}$ are two sequences defined by:

$$P_{-1} = 0, \quad P_0 = 1, \quad P_{k+1} = b_{k+1}.P_k + P_{k-1},$$

$$Q_{-1} = 1, \quad Q_0 = 0, \quad Q_{k+1} = b_{k+1}.Q_k + Q_{k-1},$$

for $k \geq 0$.

Proof. Using mathematical induction, it is proven easily. Considering the following table which includes values of A_k, B_k and a_k we can determine converge of

$$w_d = \left[\ a_0; \overline{4, 4, \ldots, 4,\ 2a_0}\ \right]$$

for $l(d) > 4$..So, this is true for $k = 0$.
Now, we suppose that the result is true for $k < i$ and $0 < i \leq l - 1$ .Using the defined relations for $\{H_i\}$ sequence, we obtain

$$\begin{aligned} A_{k+1} &= a_{k+1}.A_k + A_{k-1} = 4\left(a_0 H_{k+1} + H_k\right) + \ \left(a_0 H_k + H_{k-1}\right) \\ &= a_0\left(4H_{k+1} + H_k\right) + \ \left(4H_k + H_{k-1}\right) \\ &= a_0 H_{k+2} + H_{k+1} \end{aligned}$$

We get the following result as follows:

$$B_{k+1} = a_{k+1}.B_k + B_{k-1} = 4H_{k+1} + H_k = H_{k+2}$$

Moreoever, since $a_l = 2a_0$, we can get obtain $A_l = 2a_0^2 H_l + 3a_0 H_{l-1} + H_{l-2}$ and $B_l = 2a_0.H_l + H_{l-1}$ for $j = l(d)$ in an easy way.

In a similar way, for the continued fraction expansion $a_0 + \sqrt{d} = [b_1, b_2, \ldots, b_n, ...\ ] = [2a_0, 4, \ldots, 4, ...\ ]$, it is easy to complete the proof.

□

3 Theorems and Results

Theorem 3.1. Let d be the square free positive integer and ℓ be a positive integer holding that $\ell \equiv 0(mod2)$ and $\ell > 1$. We assume that parametrization of d is

$$d = \frac{\gamma^2 H_\ell^2}{4} + \left(2H_\ell + \ H_{\ell-1}\right)\gamma + 5.$$

for $\gamma > 0$ positive integer. If $\gamma \equiv 1(mod4)$ positive integer then $d \equiv 2(mod4)$ and

$$w_d = \left[\frac{\gamma H_\ell}{2} + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1}, \gamma H_\ell + 4}\right]$$

hold for $\ell = \ell(d)$. Moreover, the following equations also hold:

$$\epsilon_d = \left(\frac{\gamma H_\ell^2}{2} + 2H_\ell + H_{\ell-1}\right) + H_\ell\sqrt{d}$$

$$t_d = \gamma H_\ell^2 + 4H_\ell + 2H_{\ell-1} \quad \text{and} \quad u_d = 2H_\ell$$

for ϵ_d, t_d and u_d.

Remark 3.2. We should say that the present paper has got the most general theorems for such type real quadratic fields. Also, we can obtain infinitely many values of d which correspond to $Q(\sqrt{d})$ and determine structures of such fields by using the results.

Proof. Let $\ell \equiv 0(mod2)$ and $\ell > 1$ hold. If $\ell \equiv 0(mod2)$ then we have $H_\ell \equiv 0(mod4)$, $H_{\ell-1} \equiv 1(mod4)$. Considering $\gamma \equiv 1(mod4)$ positive integer and substituting these equivalent and equations into the parametrization of d then we get $d \equiv 2(mod4)$.
Using Lemma 2.4, we put

$$w_R = \frac{\gamma H_\ell}{2} + 2 + \left[\frac{\gamma H_\ell}{2} + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1}, \gamma H_\ell + 4}\right],$$

and get

$$w_R = (\gamma H_\ell + 4) + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{\ddots + \cfrac{1}{4 + \frac{1}{w_R}}}}} = (\gamma H_\ell + 4) + \frac{1}{4} + \cdots + \frac{1}{w_R}$$

Now, by using Lemma 2.4, Lemma 2.5 and the properties of continued fraction expansion, we get

$$w_R = (\gamma H_\ell + 4) + \frac{H_{\ell-1} w_R + H_{\ell-2}}{H_\ell w_R + H_{\ell-1}},$$

by using induction and property of continued fraction expansion and the Definition 2.1 into the above equality, we obtain

$$w_R^2 - (\gamma H_\ell + 4)\, w_R - (1 + \gamma H_{\ell-1}) = 0.$$

This requires that $w_R = \frac{\gamma H_\ell}{2} + 2 + \sqrt{d}$ since $w_R > 0$. Considering Lemma 2.5, we get

$$w_d = \sqrt{d} = w_d = \left[\frac{\gamma H_\ell}{2} + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1}, \gamma H_\ell + 4}\right]$$

and $\ell = \ell(d)$. This completes the first part of theorem.

Now, we should determine ϵ_d, t_d and u_d using Lemma 2.4, we have

$$Q_1 = 1 = H_1, \quad Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 4 = H_2,$$

$$Q_3 = a_2 Q_2 + Q_1 = 4H_2 + H_1 = 17 = H_3, Q_4 = 72 = H_4, \ldots$$

This implies that $Q_i = H_i$ by using mathematical induction for $\forall i \geq 0$. On substituting these values of sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}\rangle 1$ and rearranged, we get

$$\epsilon_d = \left(\frac{\gamma H_\ell^2}{2} + 2H_\ell + H_{\ell-1}\right) + H_\ell\sqrt{d}$$

$$t_d = \gamma H_\ell^2 + 4H_\ell + 2H_{\ell-1} \quad \text{and} \quad u_d = 2H_\ell$$

for ϵ_d, t_d and u_d.
This completes the proof of Theorem 3.1. □

Corollary 3.3. Let d be the square free positive integer and ℓ be a positive integer holding that $\ell \equiv 0(mod2)$ and $\ell > 1$. We assume that parametrization of d is

$$d = \frac{H_\ell^2}{4} + 2H_\ell + \ H_{\ell-1} + 5.$$

Then, we get $d \equiv 2(mod4)$ and

$$w_d = \left[\frac{H_\ell + 4}{2}; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1}, H_\ell + 4} \right]$$

and $\ell = \ell(d)$. Moreover, we have following equalities:

$$\epsilon_d = \left(\frac{H_\ell^2}{2} + 2H_\ell + H_{\ell-1} \right) + H_\ell \sqrt{d}$$

$$t_d = H_\ell^2 + 4H_\ell + 2H_{\ell-1} \quad \text{and} \quad u_d = 2H_\ell$$

and m_d=3 for $\ell \geq 4$.

Proof. We obtain this result by subsituting γ=1 into Theorem 3.1. Now, we should prove that value of Yokoi's invariant is m_d=3 for $\ell \geq 4$. If we put t_d and u_d into the m_d and rearranged, then we obtain

$$m_d = \left[\left[\frac{u_d^2}{t_d} \right]\right] = \left[\left[\frac{4H_\ell^2}{H_\ell^2 + 4H_\ell + 2H_{\ell-1}} \right]\right]$$

In the case of $\ell = 2$, d is not square free integer.That's why $\ell = 2$ is ruled out for calculating Yokoi's invariant m_d. Since H_ℓ increasing we get,

$$4 > 4.\left(1 + \frac{4}{H_\ell} + \frac{2H_{\ell-1}}{H_\ell^2}\right)^{-1} > 3,766$$

for $\ell \geq 4$. Therefore, we obtain

$$m_d = \left[\left[\frac{4H_\ell^2}{H_\ell^2 + 4H_\ell + 2H_{\ell-1}} \right]\right] = 3$$

for $\ell \geq 4$. This completes the proof of the Corollary 3.3. Besides, in the following Table 3.1 shows some numerical examples for Corollary 3.3 as follows:

d	$\ell(d)$	m_d	w_d	ϵ_d
1462	4	3	$[38; \overline{4, 4, 4, 76}]$	$2753 + 72\sqrt{1462}$
134426310	8	3	$[11594; \overline{4, 4, 4, 4, 4, 4, 4, 23188}]$	$268800769 + 23184\sqrt{134426310}$

Table 3.1:

In the above Table, fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is m_d for $2 < \ell(d) < 10$ (In this table, we have to rule out $\ell(d) = 2, 6, 10$ since d is not a square free positive integer in these periods).

□

Remark 3.4. In the Table 3.1, we calculate class number $h_d = 4$ for $Q(\sqrt{1462})$ and another class number is $h_d = 576$ for $Q(\sqrt{134426310})$, by using the classical Dirichlet class number formula.

Theorem 3.5. Let d be a square free positive integer and ℓ be a positive integer satisfying that $\ell \geq 2$. Suppose that parametrization of d is

$$d = \gamma^2 H_\ell^2 + 2\gamma(2H_\ell + H_{\ell-1}) + 5.$$

for $\gamma \geq 1$ integer. If γ is odd positive integer, we have $d \equiv 2, 3(mod4)$ and

$$w_d = \left[\gamma H_\ell + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1},\ 4 + 2\gamma H_\ell}\right]$$

with $\ell = \ell(d)$. Moreover, we can get fundamental unit ϵ_d, coefficients of fundamental unit t_d, u_d as follows:

$$\epsilon_d = (\gamma H_\ell + 2)\, H_\ell + H_{\ell-1} + H_\ell\sqrt{d},$$
$$t_d = 2\,(\gamma H_\ell + 2)\, H_\ell + 2H_{\ell-1} \quad \text{and} \quad u_d = 2H_\ell.$$

Proof. Let $\ell \geq 2$ be the positive integer. Using Remark 2.3, we get $H_\ell \equiv 1\,(mod4)$ and $H_{\ell-1} \equiv 0\,(mod4)$ for both $\ell \equiv 1\,(mod4)$ and $\ell \equiv 3\,(mod4)$.By considering γ is odd positive integer and substituting these equalities into the $d = \gamma^2 H_\ell^2 + 2\gamma(2H_\ell + H_{\ell-1}) + 5$, we get $d \equiv 2(mod4)$.

Besides, we have that $H_\ell \equiv 0\,(mod4)$ and $H_{\ell-1} \equiv 1\,(mod4)$ at the same time for both $\ell \equiv 0\,(mod4)$ and $\ell \equiv 2\,(mod4)$.By considering γ is odd positive integer and substituting these equalities into the $d = \gamma^2 H_\ell^2 + 2\gamma(2H_\ell + H_{\ell-1}) + 5$, we obtain that $d \equiv 3(mod4)$ holds.

On substituting w_d into the w_R , we get

$$w_R = (\gamma H_\ell + 2) + \left[\gamma H_\ell + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1},\ 4 + 2\gamma H_\ell}\right]$$

and we have

$$w_R = (2\gamma H_\ell + 4) + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{\ddots + \cfrac{1}{4 + \frac{1}{w_R}}}}} = (2\gamma H_\ell + 4) + \frac{1}{4} + \cdots + \frac{1}{w_R}$$

Using Lemma 2.4. and Lemma 2.5. about the properties of continued fraction expansion, we get

$$w_R = (2\gamma H_\ell + 4) + \frac{H_{\ell-1} w_R + H_{\ell-2}}{H_\ell w_R + H_{\ell-1}}$$

. and, by using Definition 2.1 into the above equality, wel obtain

$$w_R^2 - (2\gamma H_\ell + 4)\, w_R - \left(1 + 2\gamma H_{\ell-1}\right) = 0.$$

This requires that $w_R = (\gamma H_\ell + 2) + \sqrt{d}$ since $w_R > 0$. Also, using Lemma 2.5, we get

$$w_d = \sqrt{d} = \left[\gamma H_\ell + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1}, 2\gamma H_\ell + 4}\right]$$

and $\ell = \ell\,(d)$ This completes the first part of the Theorem 3.5.

Now, to determine ϵ_d, t_d and u_d using Lemma 2.4, we get

$$Q_1 = 1 = H_1, Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 4 = H_2$$

$$Q_3 = a_2 Q_2 + Q_1 = 4H_2 + H_1 = 17 = H_3, Q_4 = 72 = H_4, ...$$

So, this implies that $Q_i = H_i$ by using mathematical induction for $\forall i \geq 0$. If we substitute these values of the sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}\rangle 1$ and rearranged, we will get

$$\epsilon_d = (\gamma H_\ell + 2) H_\ell + H_{\ell-1} + H_\ell\sqrt{d},$$

$$t_d = 2(\gamma H_\ell + 2) H_\ell + 2H_{\ell-1} \quad \text{and} \quad u_d = 2H_\ell.$$

This completes the proof of the Theorem 3.5. □

Corollary 3.6. Let d be a square free positive integer and ℓ be a positive integer satisfying that $\ell \geq 2$. Suppose that parametrization of d is

$$d = H_\ell^2 + 2(2H_\ell + H_{\ell-1}) + 5$$

Then, we have $d \equiv 2, 3(mod 4)$ and

$$w_d = \left[H_\ell + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1}, \ 4 + 2H_\ell} \right]$$

with $\ell = \ell(d)$. Additionally, we can get fundamental unit ϵ_d, coefficients of fundamental unit t_d, u_d as follows:

$$\epsilon_d = (H_\ell + 2) H_\ell + H_{\ell-1} + H_\ell\sqrt{d},$$

$$t_d = 2(H_\ell + 2) H_\ell + 2H_{\ell-1} \quad \text{and} \quad u_d = 2H_\ell.$$

Also, we have value of Yokoi's d- invariant $m_d = 1$.

Proof. This corollary is obtained by using Theorem 3.5 with taking γ=1. So, we should determine value of the Yokoi's invariant m_d. we know that $m_d = \left[\left[\frac{u_d^2}{t_d} \right]\right]$ from H. Yokoi's references. If we substitute t_d and u_d into the m_d, then we get

$$m_d = \left[\left[\frac{u_d^2}{t_d} \right]\right] = \left[\left[\frac{4H_\ell^2}{2H_\ell^2 + 4H_\ell + 2H_{\ell-1}} \right]\right] = 1,$$

since H_ℓ increasing sequence and $1,28 \leq \frac{4H_\ell^2}{2H_\ell^2 + 4H_\ell + 2H_{\ell-1}} < 2$ for $\ell \geq 2$. Therefore, we obtain $m_d = 1$ for $\ell \geq 2$ owing to definition of m_d. Besides, Table 3.2. is given as numerical illustrates where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is m_d for $2 \leq \ell(d) \leq 10$.

d	$\ell(d)$	m_d	w_d	ϵ_d
39	2	1	$[6; \overline{4, 12}]$	$25+4\sqrt{39}$
370	3	1	$[19; \overline{4, 4, 38}]$	$327+17\sqrt{370}$
5511	4	1	$[74; \overline{4, 4, 4, 148}]$	$5345+72\sqrt{5511}$
94394	5	1	$[307; \overline{4, 4, 4, 4, 614}]$	$93707+305\sqrt{94394}$
1675047	6	1	$[1294; \overline{4, 4, 4, 4, 4, 2588}]$	$1672153+1292\sqrt{1675047}$
29978210	7	1	$[5475; \overline{4, 4, ..., 4, 10950}]$	$29965967+5473\sqrt{29978210}$
537601543	8	1	$[23186; \overline{4, 4, ..., 4, 46372}]$	$537549697+23184\sqrt{537601543}$
9645446890	9	1	$[98211; \overline{4, 4, ..., 4, 196422}]$	$9645227283+98209\sqrt{9645446890}$
173074500903	10	1	$[416022; \overline{4, 4, ..., 4, 832044}]$	$173073570649+416020\sqrt{173074500903}$

Table 3.2:

□

Remark 3.7. In the Table 3.2, $Q(\sqrt{39})$ is real quadratic field with class number $h_d = 2$. This field is obtained in the table 2.1 of [11] By using the classical Dirichlet class number formula, we calculate class numbers for some values of d in the Table 3.2 as follows:

d	h_d
39	2
370	4
5511	12
94394	24
1675047	72
29978210	196

Table 3.3:

Corollary 3.8. Let d be a square free positive integer and ℓ be a positive integer satisfying that $\ell \geq 2$. Suppose that parametrization of d is

$$d = 9H_\ell^2 + 2(6H_\ell + 3H_{\ell-1}) + 5$$

Then, we have $d \equiv 2, 3(mod4)$ and

$$w_d = \left[3H_\ell + 2; \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1},\ 4 + 6H_\ell} \right]$$

with $\ell = \ell(d)$. Additionally, we can get fundamental unit ϵ_d, coefficients of fundamental unit t_d, u_d as follows:

$$\epsilon_d = (3H_\ell + 2)\, H_\ell + H_{\ell-1} + H_\ell\sqrt{d},$$

$$t_d = 2\,(3H_\ell + 2)\, H_\ell + 2H_{\ell-1} \quad \text{and} \quad u_d = 2H_\ell.$$

Also, we have Yokoi's d- invariant value $n_d = 1$

Proof. Corollary is obtained by using Theorem 3.3 for γ=3. Besides, we know that $n_d = \left[\left[\frac{t_d}{u_d^2} \right]\right]$ from H. Yokoi's references. If we substitute t_d and u_d into the n_d, then we get

$$n_d = \left[\left[\frac{t_d}{u_d^2} \right]\right] = \left[\left[\frac{6H_\ell^2 + 4H_\ell + 2H_{\ell-1}}{4H_\ell^2} \right]\right] = 1,$$

since H_ℓ increasing sequence and $1,5 < \frac{6H_\ell^2+4H_\ell+2H_{\ell-1}}{4H_\ell^2} < 1,781$ for $\ell \geq 2$. Therefore, we obtain $n_d = 1$ for $\ell \geq 2$ owing to definition of n_d. Besides, Table 3.4 can be given as numerical examples for Corollary 3.8 as follows where fundamental unit is ϵ_d, integral basis element is w_d and and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 7$.

d	$\ell(d)$	n_d	w_d	ϵ_d
203	2	1	$[14; \overline{4, 28}]$	$57+4\sqrt{203}$
2834	3	1	$[53; \overline{4, 4, 106}]$	$905+17\sqrt{2834}$
47627	4	1	$[218; \overline{4, 4, 4, 436}]$	$15713+72\sqrt{47627}$
841322	5	1	$[917; \overline{4, 4, 4, 4, 1834}]$	$279757+305\sqrt{841322}$
15040715	6	1	$[3878; \overline{4, 4, 4, 4, 4, 7756}]$	$5010681+1292\sqrt{15040715}$
269656994	7	1	$[16421; \overline{4, 4, 4, 4, 4, 4, 32842}]$	$89873425+5473\sqrt{269656994}$

Table 3.4:

□

Remark 3.9. In the Table 3.4, $Q(\sqrt{203})$ is real quadratic field with class number $h_d = 2$ and the field is obtained in the Table 2.1 of [11]. In the following table, we calculate class numbers for other values of d in the Table 3.4, by using the classical Dirichlet class number formula.

d	h_d
203	2
2834	4
47627	8
841322	40
15040715	220

Table 3.5:

4 Algebraic Number Theory in Computer Science

We can see some mathematics ' topics such as Cryptology, Probability and Game Theory, Mathematical Logic, Number Theory, Individually Paced Math Sequence related with Computer Science's topics likes Fundamentals of Computer Science, Data Structures and Algorithms, Theory of Computation, Advanced Robotics...etc... Algebraic Number Theory has been greatly affected by the use of huge computing devices. There are several different ways in which computers have focused in a variety of branches of this topic. Some of them weighed are: Factorization, Continued Fraction Expansion,Primality Testing, the Syracuse Problem, Pell Equations, Abel's Problem, Diophantine Equations, Fermat's Last Theorem, Class Number Problem, the Twin Prime Conjecture, the Riemann Hypothesis, and many different problems from algebraic number theory. These topics have several applications from cryptography, to computational biology, to the study of discrete non-linear PDE... They are basically useful each time you have to solve some system of multi-variate polynomials.The study of finite groups (and also some number theory) can be useful to solve combinatorial problems which sometimes arise in theory or trying to solve some real software problem. In this paper, continued fraction expansion, fundamental units and class numbers can be useful for cryptology related with computer science.

5 Conclusion

Results obtained in this paper provide us a practical method so as to rapidly determine continued fraction expansion of w_d, fundamental unit ϵ_d and Yokoi invariants n_d ,m_d as well as class numbers for classified such real quadratic number fields. We hope that readers can use this paper in the many different fields of mathematics such as algebraic number theory, algebraic geometry, algebra, cryptology, and also other scientific fields like computer science.

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