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Note on P_{-105} Diophantine Triples

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Abstract

In this paper, we consider several P_{-105} – Diophantine triple sets. We prove that these sets aren't extended to P_{-105} – Diophantine quadruple sets even they are regular P_{-105} – Diophantine triple sets. Further, we demonstrate some common properties for the general P_{-105} – Diophantine sets using algebraic number theory and the references.

The one difficult thing for calculating diophantine triples that you have to try too many variables. This is too hard and taking too much time. For this reason, we decide to use computer science to find the diophantine triples because of the no suitable software. So, we start coding our own software during coding the software and we pay attention to some elements as follows:

- ✓ To find the right triples, we had to test all the numbers in the range.
- ✓ Because of all numbers have to be processed with each others, we use 3 loops in the each others.
- ✓ In each loop, we calculated the intermediate values.
- ✓ In the innermost loop, we calculated the result value.
- ✓ In the last step, we compared the values found and wrote the values that confirm the equation on the screen.

Keywords: *P_{-105} Diophantine Triple, Quadratic Residue Law, Quadratic Reciprocity Law, Legendre and Jacobi Symbols in Number Theory.*

1. Introduction and Preliminaries

One of the special type of P_{-105} – Diophantine triple sets is a subset of the set of natural numbers.(In thıs paper , we consider this type.) It includes special property for the definition which is defined as follows. We can also say that the concept of general Diophantine set, apparently belonging to number theory, Even it is very old problem, there are many publıshed papers and open problems in the topic. As we do in the other topics of the number theory, we again do similar in this topic and the paper. It means that our aim is to find out construction for such sets.

Many generalizations of Diophantine sets and results on the structure, readers can see the references [1- 21].

To understand paper easily, readers can find some notations as follows:

Definition 1.1.

Let $\boldsymbol{K} = \{\boldsymbol{k_1}, \boldsymbol{k_2}, \dots, \boldsymbol{k_t}\}$ be a subset (with *t* positive numbers) of the set of positive integers. $\boldsymbol{K}$ is named as Diophantine $\boldsymbol{t}$ -tuple with the property $\boldsymbol{P_s}$ over the integer numbers ring if $\boldsymbol{k_i k_j + s}$ is a perfect square integer for $\boldsymbol{i, j = 1, \dots, t}$ and $\boldsymbol{i \neq j}$.

if t=3, then it is said that this is a $\boldsymbol{P_s}$ −Diophantine triple.

Definition 1.2.

A $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is called $\boldsymbol{P_s}$- Regular Diophantine Triple if $\boldsymbol{P_s}$- Diophantine triple $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ satisfies the following condition;

$$(\boldsymbol{c} - \boldsymbol{b} - \boldsymbol{a})^2 = \boldsymbol{4}(\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{s}) \qquad (1.1)$$

The concept of the quadratic residue is important in the number theory. We also use it in our paper to get properties for such sets.

We can remind it as follows:

Definition 1.3.

if there exist an integer y such that $y^2 \equiv b \ (mod\ t)$ satisfies for $t \in \mathbb{N}$, $b \in \mathbb{Z}$ then b is called as quadratic residue $(mod\ t)$, Otherwise, it is called by quadratic non-residue.

Definition 1.4.

Assume that $b \in \mathbb{Z}$ is an integer and $q > 2$ is a prime number. Then, $\left(\frac{b}{q}\right)$ is named as Legendre Symbol . It is defined as follows.

$$\left(\frac{b}{q}\right) = \begin{cases} 1 & \text{if } b \text{ is quadratic residue modulo } q \\ -1 & \text{if } b \text{ is quadratic nonresidue modulo } q \\ 0 & \text{if } q|b \end{cases}$$

Theorem 1.1. (The Reciprocity Law.)

Let $P \neq Q$ be distinct odd primes. Then, following equation is known as the reciprocity law

$$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = (-1)^{\frac{P-1}{2} \cdot \frac{Q-1}{2}}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is Legendre symbol.

2. Main Results

Theorem 2.1. The set $P_{-105} = \{1.114.121\}$ is regular P_{-105} −Diophantine triple even if it is not extended to P_{-105} –Diophantine quadruple.

Proof. Using regularity condition defined in Definition 1.2, it is seen that $P_{-105} = \{1.114.121\}$ is a regular triple. Let $\{1.114.121\}$ be an extended to P_{-105}- Diophantine quadruple as $\{1.114.121.r\}$ for $r \in \mathbb{N}$. Then, there are $x_1.x_2.x_3$ integers and followings are satisfied;

$$r - 105 = {x_1}^2 \qquad (2.1)$$

$$114r - 105 = x^2 \qquad (2.2)$$

$$121r - 105 = {x_3}^2 \qquad (2.3)$$

If we remove r between (2.1) and (2.3), we get

$${x_3}^2 - 121{x_1}^2 = 12600 \qquad (2.4)$$

Similarly, cutting out r between (2.1) and (2.2), also (2.2) and (2.3), followings are got.

$$x_2{}^2 - 114x_1{}^2 = 11865 \quad (2.5)$$

$$114x_3{}^2 - 121x_2{}^2 = 735 \quad (2.6)$$

Considering (2.4), (2.5) and (2.6), we can seethat there is no common integer solution between them. This shows that (2.4)-(2.5)-(2.6) equation system can not have solutions in the set of integers. This is a contradiction.

Hence, $P_{-105} = \{1.114.121\}$ can not be extended to P_{-105} −Diophantine 4-tuple.

Theorem 2.2. A set $P_{-105} = \{1.121.130\}$ is both regular and non-extendible to P_{-105}-Diophantine quadruple.

Proof. If we consider (1.1), it is seen that $P_{-105} = \{1.121.130\}$ is a regular P_{-105}-Diophantine triple. By taking into consideration $\{1.121.130\}$, we can assume that the set is extended to dıophantine quadruple $\{1.121.130.n\}$ for $n \in \mathbb{N}$. So, there are $y_1.y_2.y_3$ integers so that;

$$n - 105 = y_1{}^2 \quad (2.7)$$

$$121n - 105 = y_2{}^2 \quad (2.8)$$

$$130n - 105 = y_3{}^2 \quad (2.9)$$

By removing n between (2.7) and (2.9), we have

$$y_3{}^2 - 130y_1{}^2 = 13545 \quad (2.10)$$

and doing same for n between (2.7) - (2.8) with (2.8) - (2.9), we obtain

$$y_2{}^2 - 121y_1{}^2 = 12600 \quad (2.11)$$

$$121y_3{}^2 - 130y_2{}^2 = 945 \quad (2.12)$$

Some of the integer solutions of the system (2.10), (2.11), (2.12) can be obtained as follows:

$$(y_2.y_1) = (\mp 1577.\mp 143).(\mp 359.\mp 31).(\mp 125.\mp 5).\dots\dots$$

$$(y_3.y_1) = (\mp 2215.\mp 194).(\mp 1555.\mp 136).(\mp 1305.\mp 114).\dots$$

and

$$(y_3.y_2) = (\mp 255.\mp 246).(\mp 5.\mp 4),\dots$$

If we write all integer solutions and evaulate them, we can see that system doesn’t have common solution in the set of integers. This is a contradiction.

Therefore, $P_{-105} = \{1.121.130\}$ is both regular and non-extended to P_{+55} −Diophantine4-tuple.

Theorem 2.3. A set $\{1.154.169\}$ is a regular, P_{-105} − Diophantine triple but can not be extended to P_{-105} − Diophantine quadruple.

Proof. From the regularity condition (1.1), it is easily seen that $\{1.154.169\}$ is a regular P_{-105} −Diophantine triple. Supposing that $\{1.154.169.p\}$ be a P_{-105} −Diophantine quadruple for p positive integer. Thus, we can choose $z_1.z_2.z_3$ in the set of integers such that

$$p - 105 = z_1{}^2 \quad (2.13)$$

$$81p - 105 = {z_2}^2 \tag{2.14}$$

$$146p - 105 = {z_3}^2 \tag{2.15}$$

By dropping p between (2.13) and (2.14), (2.13) and (2.15), (2.14) and (2.15), we obtain

$$-169{z_1}^2 + {z_3}^2 = 17640 \tag{2.16}$$

$$-154{z_1}^2 + {z_2}^2 = 16065 \tag{2.17}$$

$$154{z_3}^2 - 169{z_2}^2 = 1470 \tag{2.18}$$

If we search integer solutions of the (2.18), we don't get any solutions in the set of integers. It gives that there is not any integer solution of the system (2.16)-(2.17)-(2.18). So, this is a contradiction.

Therefore, $P_{-105} = \{1.154.169\}$ Diophantine triple can not be extended to P_{-105} −Diophantine 4-tuple.

Theorem 2.4. A set $\{\mathbf{1, 169, 186}\}$ is a regular $\boldsymbol{P_{-105}}$ − Diophantine triple but can not be extended to $\boldsymbol{P_{-105}}$ − Diophantine quadruple.

Proof. We can prove Theorem 2.4 by using them method above mentioned.

Theorem 2.5. Sets $\{\mathbf{1, 330, 361}\}$ and $\{\mathbf{1, 361, 394}\}$ are regular $\boldsymbol{P_{-105}}$ − Diophantine triple sets. Also, they can not be extended to $\boldsymbol{P_{-105}}$ − Diophantine quadruple.

Proof. In a similar way of the above theorem's proofs, it is seen that they are regular and non extendinle to $\boldsymbol{P_{-105}}$ − Diophantine 4-tuple.

Theorem 2.6. There isn't any integer includes multiplier 4 in the P_{-105}-Diophantine sets.

Proof. Supposing that t and $4l$ be integers in the P_{-105}-Diophantine sets for any t integer . From the definition of the P_{-105}-Diophantine sets, we get

$$4lt - 105 = L^2 \tag{2.19}$$

(2.19) implies that it has a solution for some L integers. If we apply (modulo 4) we get,

$$L^2 \equiv 3(mod\ 4) \tag{2.20}$$

(2,20) has solution if $\left(\frac{3}{4}\right) = 1$. If we consider definition of the Legendre symbol, we obtain $\left(\frac{3}{4}\right) = -1$. So, there isn't any integer satisfies $L^2 \equiv 3(mod\ 4)$ and there is not any P_{-105}-Diophantine sets include such integers.

Theorem 2.7. There is no integer contains multiplier 9 in the P_{-105}-Diophantine m-tuples.

Theorem 2.8. There is no integer includes multiplier 17, 23, 25, 29, 37, … in the P_{-105}-Diophantine m-tuples.

Proof. It will be enough to prove Theorem 2.8 for the integer 17 since others can be proven in a similar way.

Let w and $17d. (d \in \mathbb{Z})$ be positive integers in the P_{-105}-Diophantine sets. Applying definition of the P_{-105}-Diophantine sets, we have

$$17dw - 105 = D^2 \tag{2.21}$$

for integer D. Applying (mod 17) on the (2.21), we have

$$D^2 \equiv 14\ (mod\ 17) \quad (2.22)$$

Using Ledendre symbol's properties on (2.22) , we obtain

$$\left(\frac{14}{17}\right) = \left(\frac{2}{17}\right)\left(\frac{7}{17}\right) \quad (2.23)$$

Using Theorem 1.1 and Definitions from the preliminaries section, we get following results;

$$\left(\frac{2}{17}\right) = +1 \quad \text{and} \quad \left(\frac{7}{17}\right) = -1$$

If we put them in to the (2.23), we can see that there is a contraction with hyphothesis. Hence, there is no set P_{-105}-Diophantine sets contain such sets.

Conclusion

- ✓ One may extend all theorems proven can prove them by using different methods. Also, general results can be obtained for P_{-105}-Diophantine sets.
- ✓ It's fast and efficent to calculate equation with too many numbers.
- ✓ Testing equations with too many numbers by software, we have to have powerful computer.

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