A Note on the Some Specific Diophantine D(s) Property from Triple to Quadruple

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Abstract

In this paper, we consider some of the Diophantine D(400) triples (several of them are determined in the previous work (ref. [27]) and expand them to positive integer Diophantine D(400) quadruples by using techniques of Diophantine equations. Obtained results in this work are significant to see how techniques are applied and what kind of new knowledge in Diophantine theory is obtained for literature. Besides, one of the important things is to get more general theoretical results as mathematical and then consider computer programme/software for them as practically. So, software can be prepared for this special type to get some numerical results easily which lends credence to our results for large numbers.

Keywords: Diophantine D(s) Set, Solutions of Diophantine Equations Systems, Positive Integers, Regularity of Diophantine D(s) quadruples.

1. Introduction and Mathematical Background

Let t be an integer and $\{\omega_1, \omega_2, \dots \dots, \omega_r\}$ be a set of positive integers. Such set is called by property D(*t*) if $\omega_i \omega_j + t$ is a perfect square integer for all $1 \leq i \neq j \leq m$. The problem which found out by Diophantus is important and still have potential to consider and use in the literature. As a famous problem of the Diophantine D(s) triples can be given by Pythagorean Triples.

By emphasis on methods of Stolt' fundamental method, Matthew's method, Lagrange' method, Nagell fundamental solutions technique, Florida transformations method, Hermite-Serret algorithm etc..., some new specific results for Diophantine D(400) quadruples are obtained.

A survey of Diophantine D(t) sets and methods of the Diophantine equations as well as related topics could be found in (ref [1-33]).

The purpose of this work is to consider the problem of extendibility of the Diophantine D(400) triples (several of them are determined in our previous work ref.[27]). Also, it is demonstrated that such triples can be extended to Diophantine D(400) quadruples with some positive integers by the use of Stolt's, Lagrange and Mathew's Diophantine method. To give some numerical calculations and theories for some Diophantine D(400) quadruples, techniques of number theory and computer software program are used in the work. One may also consider other types of Diophantine D(400) quadruples or Diophantine D(*t*) quadruples for integers *t*.

Followings results can be given from the literature as a summary:

Definition 1. (Diophantine D(t) quadruple) Let t be a positive integer. A set of s positive integers $\Lambda = \{r_1, r_2, r_3, r_4\}$ is named by Diophantine quadruple with property D(t) if (ri.rj + t) is a perfect square integer for all i < j from 1 to 4.

Regularity Condition for Quadruples from Dujella's papers: A Diophantine *D(k)* - quadruple $\Lambda = \{r_1, r_2, r_3, r_4\}$ in the set of the positive integers is named by regular if the condition $k.(r_4 + r_3 - r_2 - r_1)^2 = 4.(r_1 r_2 + k)(r_3 r_4 + k)$ is proved.

Method of the paper: For used techniques in this work (related between fundamental solution of the Diophantine equation/binary quadratic form $\mathcal{A}u^2 + \mathcal{B}uv + \mathcal{C}v^2 = \mathcal{N}$ and the solutions of the equation $u^2 - dv^2 = \mathcal{N}$, $\mathcal{N} > 0$ or $\mathcal{N} < 0$ by using the solutions of classical Pell equation's $s^2 - dr^2 = 1$ or $s^2 - dr^2 = 4$), especially some references such as (ref. [18,21-22,26,27, 31-32]) can be offered to readers.

2. Main Results

As mentioned at the beginning of the paper, this paper aims to get extensions as Diophantine D(400) quadruple of some Diophantine D(400) triples which are obtained in the paper "Some Results on the Extensions of Special Diophantine D(s) Sets from Single to Triples" (ref. [27]). Additionally (in this paper), some new Diophantine D(400) triples with large numbers are proposed and extended to the Diophantine D(400) quadruples. Besides, the author will extend some of them to Diophantine D(400) quadruples using Stolt's, Mathews', Lagrange's techniques.

Choosing some of the Diophantine D(400) triples from the Table 1-Table 11 from the reference [27], following results will be proved for Diophantine D(400) quadruples.

Theorem 1. The set {2} is extended to some Diophantine *D(400)* quadruples with positive integers given by following Table 1.

{2, 42, 192, 600}	{2, 42, 600, 1600}	{2, 192, 250, 1848}

Table 1. Extendibility of single set {2} for Diophantine D(400) quadruples till positive integer 2500.

Proof. In the Theorem 1/Table1 and Theorem5/Table7 (ref.[27]), it is proved that $\{2, 42\}$ and $\{2, 192\}$ are Diophantine D(400) pairs and also, $\{2, 42, 192\}$, $\{2, 42, 600\}$, $\{2, 192, 250\}$ are Diophantine D(400) triples, respectively.

Let us consider Diophantine D(400) triple $\{2, 42, 192\}$ and prove how we can extend it to Diophantine D(400) quadruple as {2, 42, 192, 600}. In a similar way, others can be proved easily with same method.

Let ДК be the fourth positive integer in the set of $\{2, 42, 192, ДК\}$ as Diophantine *D(400)* four-tuples. Using definition of the Diophantine D(s) quadruple, it is said that we have following equations with solutions (A, B, C are integers);

$2ДК + 400 = A^2$, $42ДК + 400 = B^2$, $192ДК + 400 = C^2$

By dropping ДК from the all equations mentioned as above, following three Pell / Pell like equations are obtained:

$21A^2 - B^2 = 8000$ and $96A^2 - C^2 = 38000$ and $32B^2 - 7C^2 = 10000$

It is obtained following results if it is searched fundamental solutions of these Pell/Pell like equations, respectively;

So, we can use Keith R. Matthews' method with solutions of classical pell equations to solve Pell equations.

For $21A^2 - B^2 = 8000$ with discriminant value is 84. (The diophantine equation $21A^2 - B^2 = 8000$ has 12 solution families) we have fundamental solution families as follows while least positive solution of $U^2 - 84V^2 = 4$ is (φ_1,ψ_1)=(110,12)

Fundamental solution [0]: (-28, 92), A = (-28 U + 184 V)/2, B = (92 U - 1176 V)/2

Fundamental solution [1]: (28, 92), A = (28 U + 184 V)/2, B = (92 U + 1176 V)/2

Fundamental solution [2]: (-92, 412), A = (-92 U + 824 V)/2, B = (412 U - 3864 V)/2

Fundamental solution [3]: (92, 412), A = (92 U + 824 V)/2, B = (412 U + 3864 V)/2

Fundamental solution [4]: (-20, 20), A = (-20 U + 40 V)/2 , B = (20 U - 840 V)/2

Fundamental solution [5]: (20, 20), A = (20 U + 40 V)/2, B = (20 U + 840 V)/2

Fundamental solution [6]: (-60, 260), A = (-60 U + 520 V)/2 , B = (260 U - 2520 V)/2

Fundamental solution [7]: (60, 260), A = (60 U + 520 V)/2 , B = (260 U + 2520 V)/2

Fundamental solution [8]: (-24, 64), A = (-24 U + 128 V)/2 , B = (64 U - 1008 V)/2

Fundamental solution [9]: (24, 64), A = (24 U + 128 V)/2 , B = (64 U + 1008 V)/2

Fundamental solution [10]: (-40, 160), A = (-40 U + 320 V)/2, B = (160 U - 1680 V)/2

Fundamental solution [11]: (40, 160), A = (40 U + 320 V)/2, B = (160 U + 1680 V)/2

where $U^2 - 84V^2 = 4$.There are 12 families of solutions for $21A^2 - B^2 = 8000$.

For $96A^2 - C^2 = 38000$ with discriminant value is 384,there are 16 families of solutions and we have fundamental solution families as follows while least positive solution of $U^2 - 384V^2 = 4$ is (φ_1,ψ_1)=(98,5).

Fundamental solution [0]: (-48, 428) , A = (-48 U + 856 V)/2 , C = (428 U - 9216 V)/2

Fundamental solution [1]: (-26, 164), A = (-26 U + 328 V)/2, C = (164 U - 4992 V)/2

Fundamental solution [2]: (22, 92), A = (22 U + 184 V)/2, C = (92 U + 4224 V)/2

Fundamental solution [3]: (64, 596), A = (64 U + 1192 V)/2, C = (596 U + 12288 V)/2

Fundamental solution [4]: (-64, 596), A = (-64 U + 1192 V)/2, C = (596 U - 12288 V)/2

Fundamental solution [5]: (-22, 92), A = (-22 U + 184 V)/2, C = (92 U - 4224 V)/2

Fundamental solution [6]: (26, 164), A = (26 U + 328 V)/2, C = (164 U + 4992 V)/2

Fundamental solution [7]: (48, 428), A = (48 U + 856 V)/2, C = (428 U + 9216 V)/2

Fundamental solution [8]: (-20, 20), A = (-20 U + 40 V)/2, C = (20 U - 3840 V)/2

Fundamental solution [9]: (-90, 860), A = (-90 U + 1720 V)/2, C = (860 U - 17280 V)/2

Fundamental solution [10]: (30, 220), A = (30 U + 440 V)/2, C = (220 U + 5760 V)/2

Fundamental solution [11]: (-40, 340), A = (-40 U + 680 V)/2, C = (340 U - 7680 V)/2

Fundamental solution [12]: (40, 340), A = (40 U + 680 V)/2, C = (340 U + 7680 V)/2

Fundamental solution [13]: (-30, 220), A = (-30 U + 440 V)/2, C = (220 U - 5760 V)/2

Fundamental solution [14]: (90, 860), A = (90 U + 1720 V)/2, C = (860 U + 17280 V)/2

Fundamental solution [15]: (20, 20) , A = (20 U + 40 V)/2 , C = (20 U + 3840 V)/2

where $U^2 - 384V^2 = 4$.The diophantine equation $96A^2 - C^2 = 38000$ has 16 solution families.

For $32B^2 - 7C^2 = 10000$ with discriminant value is 896,there are 5 families of solutions and we have fundamental solution families as follows while least positive solution of $U^2 - 896V^2 = 4$ is (φ_1,ψ_1)=(30,1).

Fundamental solution [0]: (22, 28) , B = (22 U + 392 V)/2 , C = (28 U + 1408 V)/2

Fundamental solution [1]: (-22, 28) , B = (-22 U + 392 V)/2 , C = (28 U - 1408 V)/2

Fundamental solution [2]: (-20, 20), B = (-20 U + 280 V)/2 , C = (20 U - 1280 V)/2

Fundamental solution [3]: (20, 20) , B = (20 U + 280 V)/2, C = (20 U + 1280 V)/2

Fundamental solution [4]: (50, 100), B = (50 U + 1400 V)/2 , C = (100 U + 3200 V)/2

where $U^2 - 896V^2 = 4$. The diophantine equation $32B^2 - 7C^2 = 10000$ has 5 solution families.

When the solutions are examined, it is seen that there is a solution (A, B, C) = ($\pm 40, \pm 160, \pm 340$) provides all three Pell equations. Therefore, ДК = 600 and {2, 42, 192, 600} becomes Diophantine *D(400)* quadruple.

From the Definition of Regularity condition for Diophantıne D(s) quadruple), ıt is easily seen that $\{2, 42, 192, 600\}$ is not regular Diophantine *D(400)* quadruple.

Using similar way, we can get other Diophantine D(400) quadruples as {2, 42, 600, 1600} and {2, 192, 250, 1848}.

So, the proof is completed.

Remark 1: it is quite difficult to identify these sets as the numbers in the sets get larger. For this reason, to not have a practical algorithm is still a problem for such sets

Theorem 2. The set $\{3\}$ is extended to some Diophantine *D(400)* quadruples with positive integers given by following Table 2.

{3, 28, 75, 275}	{3, 28, 483, 1275}	{3, 147, 187, 1500}
{3, 28, 275, 483}	{3, 43, 252, 875}	{3, 187, 252, 875}
{3, 28, 275, 768}	{3, 75, 128, 700}	{3, 348, 427, 1547}

Table 2. Extendibility of single set {3} for Diophantine D(400) quadruples till positive integer 2000.

Proof. In the Theorem 2/Table2 and Theorem6/Table8 (ref.[27]), the sets $\{3, 28\}$, $\{3, 43\}$, $\{3, 75\}$, $\{3, 147\}$, $\{3, 187\}$, $\{3, 348\}$ are Diophantine D(400) pairs and also, the sets $\{3, 28, 75\}$,$\{3, 28, 275\}$,$\{3, 28, 483\}$,$\{3, 43, 252\}$,$\{3, 75, 128\}$, $\{3, 147, 187\}$, $\{3, 187, 252\}$ and {3, 348, 427} are Diophantine D(400) triples were given respectively.

Choosing one of the Diophantine D(400) triples mentioned above it can be proved extendibility to Diophantine D(400) quadruple. Let $\{3, 75, 128\}$ be a Diophantine D(400) triple, we can extend it as $\{3, 75, 128, 700\}$ Diophantine D(400) quadruple as follows:

Let $\P$ be the fourth positive integer in the set of $\{3, 75, 128, \P\}$ as Diophantine *D(400)* quadruple. From the definition of Diophantine D(s) quadruple, following equations are obtained with their solutions (M, N, R are integers);

$$3\P + 400 = M^2, \quad 75\P + 400 = N^2, \quad 128\P + 400 = R^2$$

Eliminating $\P$ from the all equations mentioned as above, we obtain following three Pell / Pell like equations;

$$25M^2 - N^2 = 9600 \quad \text{and} \quad 128M^2 - 3R^2 = 50000, \quad 128N^2 - 75R^2 = 21200$$

Following results are also obtained if it is searched fundamental solutions of these Pell/Pell like equations using similar method of the previous proof of the Theorem 1.

For $25M^2 - N^2 = 9600$ with discriminant value is 100.This gives finitely many solutions due to factorising the right side of the equation as follows:

solution[0]: (97,-475) , solution[1]: (35,-145) , solution[2]: (50,-230)

solution[3]: (22,-50) , solution[4]: (28,-100) , solution[5]: (20,20)

solution[6]: (20,-20) , solution[7]: (28,100) , solution[8]: (22,50)

solution[9]: (50,230) , solution[10]: (35,145), solution[11]: (97,475)

solution[12]: (-97,475) , solution[13]: (-35,145) , solution[14]: (-50,230)

solution[15]: (-22,50) ,solution[16]: (-28,100) ,solution[17]: (-20,-20)

solution[18]: (-20,20) , solution[19]: (-28,-100) , solution[20]: (-22,-50)

solution[21]: (-50,-230) . solution[22]: (-35,-145) , solution[23]: (-97,-475)

For $128M^2 - 3R^2 = 50000$ with discriminant value is 1536, there are 24 families of solutions and we have fundamental solutions as follows while least positive solution of $U^2 - 1536V^2 = 4$ is (φ_1,ψ_1)=(9602,245).

Fundamental solution [0]: (349, 2276) , M = (349 U + 13656 V)/2 , R = (2276 U + 89344 V)/2

Fundamental solution [1]: (-38, 212) , M = (-38 U + 1272 V)/2 , R = (212 U - 9728 V)/2

Fundamental solution [2]: (31, 156) , M = (31 U + 936 V)/2 , R = (156 U + 7936 V)/2

Fundamental solution [3]: (-272, 1772) , M = (-272 U + 10632 V)/2 , R = (1772 U - 69632 V)/2

Fundamental solution [4]: (272, 1772), M = (272 U + 10632 V)/2 , R = (1772 U + 69632v)/2

Fundamental solution [5]: (-31, 156) , M = (-31 U + 936 V)/2 , R = (156 U - 7936 V)/2

Fundamental solution [6]: (38, 212) , M = (38 U + 1272 V)/2 , R = (212 U + 9728 V)/2

Fundamental solution [7]: (-349, 2276) , M = (-349 U + 13656 V)/2 , R = (2276 U - 89344 V)/2

Fundamental solution [8]: (115, 740) , M = (115 U + 4440 V)/2 , R = (740 U + 29440 V)/2

Fundamental solution [9]: (830, 5420), M = (830 U + 32520 V)/2 , R = (5420 U + 212480 V)/2

Fundamental solution [10]: (-85, 540) , M = (-85 U + 3240 V)/2 , R = (540 U - 21760 V)/2

Fundamental solution [11]: (20, 20) , M = (20 U + 120 V)/2 , R = (20 U + 5120 V)/2

Fundamental solution [12]: (-20, 20) , M = (-20 U + 120 V)/2 , R = (20 U - 5120 V)/2

Fundamental solution [13]: (85, 540) , M = (85 U + 3240 V)/2, R = (540 U + 21760 V)/2

Fundamental solution [14]:(-830, 5420) M = (-830 U + 32520 V)/2 R = (5420U - 212480 V)/2

Fundamental solution [15]: (-115, 740), M = (-115 U + 4440 V)/2, R = (740 U - 29440 V)/2

Fundamental solution [16]: (-25, 100), M = (-25 U + 600 V)/2, R = (100 U - 6400 V)/2

Fundamental solution [17]: (50, 300) , M = (50 U + 1800 V)/2 , R = (300 U + 12800 V)/2

Fundamental solution [18]:(-475, 3100) , M = (-475 U + 18600 V)/2, R = (3100 U - 121600 V)/2

Fundamental solution [19]: (-200, 1300), M = (-200 U + 7800 V)/2 , R = (1300 U - 51200 V)/2

Fundamental solution [20]: (200, 1300), M = (200 U + 7800 V)/2, R = (1300 U + 51200 V)/2

Fundamental solution [21]: (475, 3100), M= (475 U + 18600 V)/2, R = (3100 U + 121600 V)/2

Fundamental solution [22]: (-50, 300), M = (-50 U + 1800 V)/2, R = (300 U - 12800 V)/2

Fundamental solution [23]: (25, 100), M = (25 U + 600 V)/2, R = (100 U + 6400 V)/2

where $U^2 - 1536V^2 = 4$. The diophantine equation $128M^2 - 3R^2 = 50000$ has 24 solution families.

For $128N^2 - 75R^2 = 21200$ with discriminant value is 38400,there are 8 families of solutions and we have fundamental solutions as follows while least positive solution of $U^2 - 38400V^2 = 4$ is (φ_1,ψ_1)=(9602,49).

Fundamental solution [0]: (25, 28) , N = (25 U + 4200 V)/2 , R = (28 U + 6400 V)/2

Fundamental solution [1]: (175, 228) , N = (175 U + 34200 V)/2 , R = (228 U + 44800 V)/2

Fundamental solution [2]: (-175, 228) , N = (-175 U + 34200 V)/2 , R = (228 U - 44800 V)/2

Fundamental solution [3]: (-25, 28) , N = (-25 U + 4200 V)/2 , R = (28 U - 6400 V)/2

Fundamental solution [4]: (230, 300) , N = (230 U + 45000 V)/2 , R = (300 U + 58880 V)/2

Fundamental solution [5]: (20, 20) , N = (20 U + 3000 V)/2 , R = (20 U + 5120 V)/2

Fundamental solution [6]: (-20, 20) , N = (-20 U + 3000 V)/2 , R = (20 U - 5120 V)/2

Fundamental solution [7]: (-230, 300) , N = (-230 U + 45000 V)/2 , R = (300 U - 58880 V)/2

where $U^2 - 38400V^2 = 4$

The diophantine equation $128N^2 - 75R^2 = 21200$ has 8 solution families.

When the solutions are examined, it is seen that there is a solution (M, N, R) = $(\pm 50, \pm 230, \pm 300)$ provides all three Pell equations. Therefore, $\text{ﭏ} = 700$ and {3, 75, 128, 700}becomes Diophantine *D(400)* quadruple.

From the Definition of Regularity condition for Diophantıne D(s) quadruple), ıt is easily seen that {3, 75, 128, 700}is not regular Diophantine *D(400)* quadruple.
Using similar way, we can get other Diophantine D(400) quadruples as {3, 28, 75, 275},{3, 28, 275, 483}, {3, 28, 275, 768}, {3, 28, 483, 1275}, {3, 43, 252, 875}, {3, 147, 187, 1500}, {3, 187, 252, 875} ,{3, 348, 427, 1547}.

Theorem 3. (a) The set {4} is extended to some Diophantine *D(400)* quadruples with positive integers given by following Table 3.

{4, 21, 69, 125}	{4, 69, 125, 741}
{4, 21, 69, 384}	{4, 69, 384, 1925}
{4, 21, 125, 384}	{4, 96, 156, 224}
{4, 21, 429, 1125}	{4, 96, 224, 1500}
{4, 69, 125, 384}	{4, 125, 189, 1581}

Table 3. Extendibility of single set {4} for Diophantine D(400) quadruples till positive integer 2500.

(b) The set {5} is extended to some Diophantine *D(400)* quadruples with positive integers given by following Table 4.

{5, 45, 100, 525}	{5, 100, 165, 525}	{5, 100, 165, 1365}

Table 4. Extendibility of single set {5}for Diophantine D(400) quadruplse till value 2500.

(c) The set {7} is extended to some Diophantine *D(400)* quadruples with positive integers given by following Table 5.

{7, 12, 63, 128}	{7, 12, 375, 975}	{7, 128, 207, 663}
{7, 12, 63, 375}	{7, 12, 975, 2432}	{7, 183, 812, 1767}
{7, 12, 128, 375}	{7, 63, 128, 375}	{7, 272, 375, 1287}

Table 5. Extendibility of single set {7} for Diophantine D(400) quadruples till positive integer 2500.

Proof. It was proved that the sets {4, 21}, {4, 69}, {4, 96}, {4, 125} are Diophantine D(400) pairs in the Theorem 3/Table 3 (ref.[27]), the set {5, 45} is Diophantine D(400) pairs in the Theorem 4/Table 4 (ref.[27]), the sets {7, 12}, {7, 63}, {7, 128}, {7, 183}, {7, 272} are Diophantine D(400) pairs in the Theorem 4/Table 5 (ref.[27]).

Besides, it was demonstrated that the sets {4, 21, 69}, {4, 21, 125}, {4, 21, 429}, {4, 69, 125}, {4, 69, 384}, {4, 96, 156}, {4, 96, 224}, {4, 125, 189} are Diophantine D(400) triple in the Theorem 7/ Table 9 (ref.[27]), the set {5, 45, 100} and {5, 100, 165} are Diophantine D(400) triples in the Theorem 7/ Table 10 (ref.[27), the sets {7, 12, 63}, {7, 12, 128}, {7, 12, 375}, {7, 12, 975} , {7, 63, 128}, {7, 128, 207}, {7, 183, 812}, {7, 272, 375} are Diophantine D(400) triples in the Theorem 7/ Table 11 (ref.[27]).

In a similar manner of the proof of previous last two theorems and methods, it is seen that Diophantine D(400) quadruples are obtained and Table 3-4-5 are prepared.

Theorem 4. **(a)** The set {25} is extended to Diophantine *D(400)* triples with positive integers given by following Table 6.

{25, 33, 128}	{25, 84, 209}	{25, 209, 384}	{25, 513, 768}	{25, 1073, 1428}	{25, 1833, 2288}
{25, 33, 273}	{25, 84, 425}	{25, 240, 425}	{ 25, 560, 825}	{25, 1140, 1505}	{25, 1920, 2385}
{25, 33, 1428}	{25, 105, 240}	{25, 273, 468}	{25, 609, 884}	{25, 1209, 1584}	{25, 2009, 2484}
{25, 33, 2900}	{25, 105, 825}	{25, 273, 825}	{25, 660, 945}	{25, 1280, 1665}	{25, 2100, 2585}
{25, 48, 153}	{25, 105, 2100}	{25, 273, 2900}	{25, 713, 1008}	{25, 1353, 1748}	{25, 2193, 2688}
{25, 48, 825}	{25, 128, 273}	{25, 308, 513}	{25, 768, 1073}	{25, 1428, 1833}	{25, 2288, 2793}
{25, 48, 2288}	{25, 128, 1428}	{25, 345, 560}	{25, 825, 1140}	{25, 1505, 1920}	{25, 2385, 2900}
{25, 65, 105}	{25, 153, 308}	{25, 384, 609}	{25, 884, 1209}	{25, 1584, 2009}	
{25, 65, 180}	{25, 153, 2288}	{25, 425, 660}	{25, 945, 1280}	{25, 1665, 2100}	
{25, 65, 2100}	{25, 180, 345}	{25, 468, 713}	{25, 1008, 1353}	{25, 1748, 2193}	

Table 6. Extendibility of {25} to Diophantine *D(400)* triples till positive integer 3000.

(b) The set {25} is extended to some Diophantine *D(400)* quadruples with positive integers given by following Table 7.

{25, 33, 128, 273}	{25, 48, 153, 2288}
{25, 33, 128, 1428}	{25, 65, 105, 2100}

Table 7. Extendibility of single set {25} for Diophantine D(400) quadruples till positive integer 2500.

Proof. **(a)** It was demonstrated that the sets {25, 33}, {25, 48}, {25, 65}, {25, 84}… {25, 2385} are Diophantine D(400) pairs in the Theorem4 /Table 6 (ref [27]).

Here, in particular, how the {25,33} Diophantine D(400) pair can be extended to a Diophantine D(400) triple and how the solutions of the Pell/Pell-like equations can be obtained are expressed briefly as follows.

From this point of view, it is stated that other pairs can be expanded by solving different pell equations with similar way to get Diophantine D(400) triples.

Let {25, 33, Ѣ} be a Diophantine D(400) triple for some Ѣ positive integers. From the definition of the Diophantine D(400) triple, we get following equations with their solutions (U, V are integers):

$$25\text{Ѣ} + 400 = U^2 \text{ and } 33\text{Ѣ} + 400 = V^2$$

By eliminating Ѣ from the equations, we have Diophantine equation as $33U^2 - 25V^2 = 3200$. If we search fundamental solutions of the Pell equation, (Discriminant = 3300), we have following 18 families of different solutions as follows;

Least positive solution of $M^2 - 3300N^2 = 4$ (φ_1,ψ_1)=(97198,1692)

Fundamental solution [0]: (-85, 97), U = (-85 M + 4850 N)/2 , V = (97 M - 5610 N)/2

Fundamental solution [1]: (85, 97), U = (85 M + 4850 N)/2, V = (97 M + 5610 N)/2

Fundamental solution [2]: (-15, 13), U = (-15 M + 650 N)/2 , V = (13 M - 990 N)/2

Fundamental solution [3]: (15, 13) , U = (15 M + 650 N)/2 , V = (13 M + 990 N)/2

Fundamental solution [4]: (-605, 695), U = (-605 M + 34750 N)/2 , V = (695 M - 39930 N)/2

Fundamental solution [5]: (605, 695) , U = (605 M + 34750 N)/2 , V = (695 M + 39930 N)/2

Fundamental solution [6]: (-10, 2), U = (-10 M + 100 N)/2, V = (2 M - 660 N)/2

Fundamental solution [7]: (10, 2) , U = (10 M + 100 N)/2 , V = (2 M + 660 N)/2

Fundamental solution [8]: (-190, 218) , U = (-190 M + 10900 N)/2, V = (218 M - 12540 N)/2

Fundamental solution [9]: (190, 218), U = (190 M + 10900 N)/2 , V = (218 M + 12540 N)/2

Fundamental solution [10]: (-270, 310) , U = (-270 M + 15500 N)/2 , V = (310 M - 17820 N)/2

Fundamental solution [11]: (270, 310) , U = (270 M + 15500 N)/2 , V = (310 M + 17820 N)/2

Fundamental solution [12]: (-60, 68) , U = (-60 M + 3400 N)/2 , V = (68 M - 3960 N)/2

Fundamental solution [13]: (60, 68), U = (60 M + 3400 N)/2 , V = (68u M + 3960 N)/2

Fundamental solution [14]: (-860, 988) , U = (-860 M + 49400 N)/2 , V = (988 M - 56760 N)/2

Fundamental solution [15]: (860, 988) , U = (860 M + 4940 N)/2 , V = (988 M + 56760 N)/2

Fundamental solution [16]: (-20, 20) , U = (-20 M + 1000 N)/2, V = (20u M - 1320 N)/2

Fundamental solution [17]: (20, 20) , U = (20 M + 1000 N)/2 , V = (20 M + 1320 N)/2

where $M^2 - 3300N^2 = 4$. The diophantine equation $33U^2 - 25V^2 = 3200$ has 18 solution families.

Using these solutions and consider (U , V)=(60, 68), we obtain Ѣ=128. So, $\{25, 33, 128\}$ is Diophantine D(400) triple.

Similarly, other Diophantine D(400) triples are obtained from $\{25\}$ to Diophantine *D(400)* single. Also, we should prove whether or not $\{25, 33, 128\}$ is Regular Diophantine *D(400)* triple?

Using regularity condition for Diophantine D(s) triples given in preliminaries section of (ref.[27]), it is seen that ($(128 - 33 - 25)^2 = 4.(25.33 + 400)$) holds and it is regular Diophantine *D(400)* triple. Hence, the proof is completed.

(b) It is proved that the sets {25, 33, 128} {25, 48, 153}, {25, 65, 105} are Diophantine D(400) triples in the previous item.

Let $\{25, 33, 128\}$ be a Diophantine D(400) triple and Ӹ be the fourth positive integer in the set of $\{25, 33, 128, \text{Ӹ}\}$ as Diophantine *D(400)* quadruple. From the definition of Diophantine D(s) quadruple, following equations are obtained with their solutions (a, b, c are integers);

$$25\text{Ӹ} + 400 = a^2 , \quad 33\text{Ӹ} + 400 = b^2 , \quad 128\text{Ӹ} + 400 = c^2$$

Dropping Ӹ from the equations, following three Pell / Pell like equations are obtained:

$$33a^2 - 25b^2 = 3200 \quad \text{and} \quad 128a^2 - 25c^2 = 41200, \quad 128b^2 - 33c^2 = 38000.$$

From previous item, solutions of the $33a^2 - 25b^2 = 3200$. was known.

For $128a^2 - 25c^2 = 41200$ with discriminant value is 12800 and there are 24 families of solutions. We have fundamental solution families as follows while least positive solution of $U^2 - 12800V^2 = 4$ is (φ_1,ψ_1)=(1536796802, 13583493).

Fundamental solution [0]: (-96985, 219452) , a = (-96985 U + 10972600 V)/2 , c = (219452 U - 24828160 V)/2

Fundamental solution [1]: (16640, 37652) , a = (16640 U + 1882600 V)/2 , c = (37652 U + 4259840 V)/2

Fundamental solution [2]: (-6440, 14572) , a = (-6440 U + 728600 V)/2 , y c = (14572 U - 1648640 V)/2

Fundamental solution [3]: (37535, 84932) , a = (37535 U + 4246600 V)/2 , c = (84932 U + 9608960 V)/2

Fundamental solution [4]: (35, 68) , a = (35 U + 3400 V)/2 , c = (68 U + 8960 V)/2

Fundamental solution [5]: (-190, 428) , a = (-190 U + 21400 V)/2 , c = (428 U - 48640 V)/2

Fundamental solution [6]: (-490, 1108) , a = (-490 U + 55400 V)/2 , c = (1108 U - 125440 V)/2

Fundamental solution [7]: (85, 188) , a = (85 U + 9400 V)/2 , c = (188 U + 21760 V)/2

Fundamental solution [8]: (-85, 188) , a = (-85 U + 9400 V)/2 , c = (188 U - 21760 V)/2

Fundamental solution [9]: (490, 1108) , a = (490 U + 55400 V)/2 , c = (1108 U + 125440 V)/2

Fundamental solution [10]: (190, 428) , a = (190 U + 21400 V)/2 , c = (428 U + 48640 V)/2

Fundamental solution [11]: (-35, 68) , a = (-35 U + 3400 V)/2 , c = (68 U - 8960 V)/2

Fundamental solution [12]: (-37535, 84932) , a = (-37535 U + 4246600 V)/2 , c = (84932 U - 9608960 V)/2

Fundamental solution [13]: (6440, 14572) , a = (6440 U + 728600 V)/2 , c = (14572 U + 1648640 V)/2

Fundamental solution [14]: (-16640, 37652) , a = (-16640 U + 1882600 V)/2 , c = (37652 U - 4259840 V)/2

Fundamental solution [15]: (96985, 219452) , a = (96985 U + 10972600 V)/2 , c = (219452 U + 24828160 V)/2

Fundamental solution [16]: (2855, 6460) , a = (2855 U + 323000 V)/2 , c = (6460 U + 730880 V)/2

Fundamental solution [17]: (218770, 495020) , a = (218770 U + 24751000 V)/2 , c = (495020 U + 56005120 V)/2

Fundamental solution [18]: (-1105, 2500) , a = (-1105 U + 125000 V)/2 , c = (2500 U - 282880 V)/2

Fundamental solution [19]: (20, 20) , a = (20 U + 1000 V)/2 , c = (20 U + 5120 V)/2

Fundamental solution [20]: (-20, 20) , a = (-20 U + 1000 V)/2 , c = (20 U - 5120 V)/2

Fundamental solution [21]: (1105, 2500) , a = (1105 U + 125000 V)/2 , c = (2500 U + 282880 V)/2

Fundamental solution[22]:(-218770, 495020),a = (-218770 U + 24751000 V)/2, c = (495020 U - 56005120 V)/2

Fundamental solution [23]: (-2855, 6460) , a = (-2855 U + 323000 V)/2 , c = (6460 U - 730880 V)/2

where $U^2 - 12800V^2 = 4$. The diophantine equation $128a^2 - 25c^2 = 41200$ has 24 solution families.

For $128b^2 - 33c^2 = 38000$ with discriminant value is 16896,there are 8 families of solutions and we have fundamental solution families as follows while least positive solution of $U^2 - 16896V^2 = 4$ is (φ_1,ψ_1)=(130, 1).

Fundamental solution [0]: (97, 188) , b = (97 U + 12408 V)/2 , c = (188 U + 24832 V)/2

Fundamental solution [1]: (-46, 84) , b = (-46 U + 5544 V)/2 , c = (84 U - 11776 V)/2

Fundamental solution [2]: (46, 84) , b = (46 U + 5544 V)/2 , c = (84 U + 11776 V)/2

Fundamental solution [3]: (-97, 188) , b = (-97 U + 12408 V)/2 , c = (188 U - 24832 V)/2

Fundamental solution [4]: (20, 20) , b = (20 U + 1320 V)/2 , c = (20 U + 5120 V)/2

Fundamental solution [5]: (-35, 60) , b = (-35 U + 3960 V)/2 , c = (60 U - 8960 V)/2

Fundamental solution [6]: (35, 60) , b = (35 U + 3960 V)/2 , c = (60 U + 8960 V)/2

Fundamental solution [7]: (-20, 20) , b = (-20 U + 1320 V)/2 , c = (20 U - 5120 V)/2

where $U^2 - 16896V^2 = 4$. The diophantine equation $128b^2 - 33c^2 = 38000$ has 8 solution families

It is seen that there is a solution $(a, b, c\ )$ = $(\pm 85, \pm 97, \pm 188)$ provides all three Pell equations. Therefore, Ӹ $= 273$ and also, {25, 33, 128, 273} is Diophantine *D(400)* quadruple.

From the Definition of Regularity condition for Diophantıne D(s) quadruple), ıt is easily seen that {25, 33, 128, 273} is not regular Diophantine *D(400)* quadruple.

Besides, other Diophantine D(400) quadruples are determined as {25, 33, 128, 273}, {25, 33, 128, 1428}, {25, 48, 153, 2288}, {25, 65, 105, 2100}.

3. Discussion and Conclusion

The main contribution of this work is to prove how we can get results from triple to quadruple and solve Diophantine equation system for some triple D(400) properties with different techniques. To calculate some large values or solutions of the Diophantine equations, a useful software can be implemented and used. Furthermore, this work is obtained both expanded of one of our previous works and new results while putting them to literature. The subject is related to Diophantine set theory/Diophantine equation theory in mathematics and also knowledge discovery. Readers may also use our results/perspective to get different results for such sets.

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