Some Results on the Extensions of Special Diophantine D(s) Sets from Single to Triples

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Abstract

Diophantine equation and Diophantine set theory have remained popular for years. They are one of the important topics in the field of number theory, which still has open problems.

In this study, it is considered how Diophantine D(s) sets of a special type (for a selected positive integer value of s) can be initialized with arbitrary positive integer single element sets. Then, they expand to three-elements of positive integers Diophantine D(s) sets. It is shown numerically using algebraic elementary operations and different solution methods of Pell's equations. These results are important to get new knowledge for Diophantine theory and useful for literature. This study is a base for us to prepare our future work too. As it is seen that it is also not easy to determine these types of sets if we face large numbers. Using practical computer algorithm (but there is still no this kind of useful and practical algorithm for each types of diophantine sets with many tuples) some results can be obtained as numerically. However, one of the important things is to have more general theoretical results firstly and then apply it to computer programme as practically.

Keywords: Diophantine D(s) Set, Characterization of D(s) Sets, Positive Integers, Methods of Diophantine Equations, Regularity of Diophantine D(s) triples.

1. Introduction

Many studies have been done on the diophantine set theory, but it is obvious that some results still cannot be obtained or some practical methods are not available for the solutions of some problems. Each of the open problems that continue to be solved with different methods contributes to the literature from different perspectives and expands the literature on the subject.

Set theory with D(n) property can be determined by Diophantine equation methods (eg. Stolt' fundamental method, Mathew's method, Lagrange' method, Nagell fundamental solutions technique, Florida transformations method, Hermite-Serret algorithm etc...).It can also be determined by a common solution with topics such as elliptic curves, Siegel's theorem, Paley graph theorem, field extensions and so on…

All the basic information and important studies related to the aforementioned methods and our work on Diophantine set theory with property D(n) are given in our references.

Pell/Diophantine equation theory has been worked by many mathematicians as Archimedes, Brahmagupta, Pierre de Fermat, John Pell, Euler etc… (ref [1-27]). Using these methods and the sources in our references, both the preliminary information section and the content of our article have been prepared.

In this paper, we consider the problem of extendibility of the singles of the form {x} where x equals to two (2) or three (3) or four (4) or … nine (9) so on… We demonstrated that {x} Diophantine D(400) single can be extended to Diophantine D(400) triples with some numerical y,z positive integers as {x, y, z} by the help of Diophantine equations' solvability with different techniques. The research question in this study is to get and find out new results for Diophantine D(400) sets as numerical. Numerical tables are also added to promote our results.

2. Mathematical Background

Followings are useful for our work and they can be given briefly from the literature:

Definition 1. (Diophantine s-tuple) Assume that s is positive integer. A set of s positive integers $\Lambda = \{r_1, r_2, r_3 \dots . r_s\}$ is named by a Diophantine s-tuple if (ri.rj + 1) is a perfect square integer for all i < j from 1 to s.

Although Euler, Fermat, Baker and Davenport got some important results on it, there are still some open questions on this work whether or not any Diophantine sextuple/ septuple or more extension as Diophantine tuple? and more…

Definition 2. (Property D(n)) Assume that s and k are positive integers. A set of s positive integers $\Lambda = \{r_1, r_2, r_3 \dots . r_s\}$ is named by Diophantine s-tuple with property D(n) if (ri.rj + k) is a perfect square integer for all i < j from 1 to s.

For example; In property D(361), it can be given quadruple as $\{44, 100, 282, 720\}$.

Regularity Condition for Triples from Dujella's papers: A Diophantine *D(k)* -triple $\Lambda = \{r_1, r_2, r_3\}$ is a set of the positive integers is named by regular if the condition $(r_3 - r_2 - r_1)^2 = 4.(r_1 r_2 + k)$ is satisfied.

(*) For the question "Assume that $\Lambda = \{r_1, r_2, r_3 \dots . r_s\}$ is a Diophantine s-tuple with property D(k). Can s be arbitrarily large?", following Siegel's theorems help us:

Theorem 1. (Siegel's Theorem) Let $\Lambda = \{r_1, r_2, r_3\}$ be a Diophantine tuple. By considering the elliptic curve $y^2 = (r_1 x + 1)(r_2 x + 1)(r_3 x + 1)$. Then, $x = r_4, r_5$, … each generate an integer point on this curve.

Theorem 2. (Siegel) The number of integer points on the elliptic curve $y^2 = x^3 + ax + b$ is finite. Hence $|\Lambda|$ is finite.

Additionally, Dujella (2001-2004) proved that there aren't any Diophantine sextuple and at most finitely many Diophantine quintuples. Togbe-Ziegler (2019) demonstrated that there aren't Diophantine quintuple.

3. Main Results

Starting from the single sets, Diophantine *D(400)* pairs will be obtained as follows:

Theorem 1. The set $\{2\}$ can be extended to Diophantine *D(400)* pairs with positive integers given by the following Table 1.

Table 1. Extendibility of $\{2\}$ to Diophantine *D(400)* pairs till 5000.

{2, 42}	{2, 312}	{2, 682}	{2, 1152}	{2, 1722}	{2, 2392}	{2, 3162}	{2, 4032}
{2, 88}	{2, 378}	{2, 768}	{2, 1258}	{2, 1848}	{2, 2538}	{2, 3328}	{2, 4218}
{2, 138}	{2, 448}	{2, 858}	{2, 1368}	{2, 1978}	{2, 2688}	{2, 3498}	{2, 4408}
{2, 192}	{2, 522}	{2, 952}	{2, 1482}	{2, 2112}	{2, 2842}	{2, 3672}	{2, 4602}
{2, 250}	{2, 600}	{2, 1050}	{2, 1600}	{2, 2250}	{2, 3000}	{2, 3850}	{2, 4800 }

Proof. Let μ be a second positive integer which makes the set $\{2\}$ as Diophantine *D(400)* pairs. Then, the following equation (a is an integer) are obtained using the definition of Diophantine *D(400)* pairs:

$$2\mu + 400 = a$$

It is seen that $\mu = 42$ is the smallest positive integer that makes the left-hand side of the equation a perfect square. Different from it, there are other many positive μ integers that make single set $\{2\}$ as Diophantine *D(400)* pairs as it is seen from Table 1.

So, the sets in the Table 1 are determined as Diophantine *D(400)* pair from the single set $\{2\}$ till positive integers 5000.

Theorem 2. The set $\{3\}$ is extended to Diophantine *D(400)* pair with positive integers given by the following Table 2.

Table 2. Extendibility of $\{3\}$ to Diophantine *D(400)* pair till 5000.

{3, 28}	{3, 323}	{3, 768}	{3, 1363}	{3, 2108}	{3, 3003}	{3, 3900}
{3, 43}	{3, 348}	{3, 803}	{3, 1408}	{3, 2163}	{3, 3068}	{3, 4048}
{3, 75}	{3, 400}	{3, 875}	{3, 1500}	{3, 2275}	{3, 3200}	{3, 4123}
{3, 92}	{3, 427}	{3, 912}	{3, 1547}	{3, 2332}	{3, 3267}	{3, 4275}
{3, 128}	{3, 483}	{3, 988}	{3, 1643}	{3, 2448}	{3, 3362}	{3, 4352}
{3, 147}	{3, 512}	{3, 1027}	{3, 1692}	{3, 2507}	{3, 3403}	{3, 4508}
{3, 187}	{3, 572}	{3, 1107}	{3, 1792}	{3, 2627}	{3, 3472}	{3, 4587}
{3, 208}	{3, 603}	{3, 1148}	{3, 1843}	{3, 2688}	{3, 3612}	{3, 4747}
{3, 252}	{3, 667}	{3, 1232}	{3, 1947}	{3, 2812}	{3, 3683}	{3, 4828}
{3, 275}	{3, 700}	{3, 1275}	{3, 2000}	{3, 2875}	{3, 3827}	{3, 4992}

Proof. Let Δ be a second positive integer which makes the set $\{3\}$ as Diophantine *D(400)* pairs. Then, the following equation (b is an integer) is obtained using the definition of Diophantine *D(400)* pairs:

$$3\Delta + 400 = b$$

It is seen that $\Delta = 28$ is the smallest positive integer that makes the left-hand side of the equation a perfect square. Additionally, there are other many positive Δ integers that make single set $\{3\}$ as Diophantine *D(400)* pairs as it is seen from Table 2.

So, the sets in the Table 2 are determined as Diophantine *D(400)* pairs from the single set $\{3\}$ till positive integers' value 5000.

Theorem 3. The set $\{4\}$ is extended to Diophantine *D(400)* pairs with positive integers given by the following Table 3.

Table 3. Extendibility of $\{4\}$ to Diophantine *D(400)* pair till 5000.

{4, 21}	{4, 300}	{4, 741}	{4, 1344}	{4, 2109}	{4, 3036}	{4, 4125}
{4, 44}	{4, 341}	{4, 800}	{4, 1421}	{4, 2204}	{4, 3149}	{4, 4256}
{4, 69}	{4, 384}	{4, 861}	{4, 1500}	{4, 2301}	{4, 3264}	{4, 4389}
{4, 96}	{4, 429}	{4, 924}	{4, 1581}	{4, 2400}	{4, 3381}	{4, 4524}
{4, 125}	{4, 476}	{4, 989}	{4, 1664}	{4, 2501}	{4, 3500}	{4, 4661}
{4, 156}	{4, 525}	{4, 1056}	{4, 1749}	{4, 2604}	{4, 3621}	{4, 4800}
{4, 189}	{4, 576}	{4, 1125}	{4, 1836}	{4, 2709}	{4, 3744}	{4, 4941}
{4, 224}	{4, 629}	{4, 1196}	{4, 1925}	{4, 2816}	{4, 3869}	
{4, 261}	{4, 684}	{4, 1269}	{4, 2016}	{4, 2925}	{4, 3996}	

Proof. Let ∂ be a second positive integer which makes the set $\{4\}$ as Diophantine *D(400)* pairs. Then, the following equation (c is an integer) is obtained using the definition of Diophantine *D(400)* pairs:

$$4\partial + 400 = c$$

It is seen that $\partial = 21$ is the smallest positive integer that makes the left-hand side of the equation a perfect square. Besides, there are other many positive ∂ integers that make single set $\{4\}$ as Diophantine *D(400)* pairs as it is seen from Table 3.

So, the sets in the Table 3 are determined as Diophantine *D(400)* pairs from the single set $\{4\}$ till the positive integers' value 5000.

Theorem 4. **(a)** The set $\{5\}$ is extended to Diophantine *D(400)* pairs with positive integers given by the following Table 4.

Table 4. Extendibility of {5} to Diophantine *D(400)* pairs up to 5000.

{5, 45}	{5, 325}	{5, 765}	{5, 1365}	{5, 2125}	{5, 3045}	{5, 4125}
{5, 100}	{5, 420}	{5, 900}	{5, 1540}	{5, 2340}	{5, 3300}	{5, 4420}
{5, 165}	{5, 525}	{5, 1045}	{5, 1725}	{5, 2565}	{5, 3565}	{5, 4725}
{5, 240}	{5, 640}	{5, 1200}	{5, 1920}	{5, 2800}	{5, 3840}	

(b) The set {7} is extended to Diophantine *D(400)* pairs with positive integers given by the following Table 5.

Table 5. Extendibility of {5} to Diophantine *D(400)* pairs up to 5000.

{7, 12}	{7, 207}	{7, 528}	{7, 975}	{7, 1548}	{7, 2247}	{7, 3072}	{7, 4023}
{7, 47}	{7, 272}	{7, 623}	{7, 1100}	{7, 1703}	{7, 2432}	{7, 3287}	{7, 4268}
{7, 63}	{7, 300}	{7, 663}	{7, 1152}	{7, 1767}	{7, 2508}	{7, 3375}	{7, 4368}
{7, 108}	{7, 375}	{7, 768}	{7, 1287}	{7, 1932}	{7, 2703}	{7, 3600}	{7, 4623}
{7, 128}	{7, 407}	{7, 812}	{7, 1343}	{7, 2000}	{7, 2783}	{7, 3692}	{7, 4727}
{7, 183}	{7, 492}	{7, 927}	{7, 1488}	{7, 2175}	{7, 2988}	{7, 3927}	{7, 4992}

(c) The set {25} is extended to Diophantine *D(400)* pairs with positive integers given by the following Table 6.

Table 6. Extendibility of {5} to Diophantine *D(400)* pairs up to 4883.

{25, 33}	{25, 209}	{25, 513}	{25, 945}	{25, 1505}	{25, 2193}	{25, 3009}	{25, 3953}
{25, 48}	{25, 240}	{25, 560}	{25, 1008}	{25, 1584}	{25, 2288}	{25, 3120}	{25, 4080}
{25, 65}	{25, 273}	{25, 609}	{25, 1073}	{25, 1665}	{25, 2385}	{25, 3233}	{25, 4209}
{25, 84}	{25, 308}	{25, 660}	{25, 1140}	{25, 1748}	{25, 2484}	{25, 3348}	{25, 4277}
{25, 105}	{25, 345}	{25, 713}	{25, 1209}	{25, 1833}	{25, 2585}	{25, 3465}	{25, 4340}
{25, 128}	{25, 384}	{25, 768}	{25, 1280}	{25, 1920}	{25, 2688}	{25, 3584}	{25, 4473}
{25, 153}	{25, 425}	{25, 825}	{25, 1353}	{25, 2009}	{25, 2793}	{25, 3705}	{25, 4608}
{25, 180}	{25, 468}	{25, 884}	{25, 1428}	{25, 2100}	{25, 2900}	{25, 3828}	{25, 4745}

Proof. Using the similar way of the previous proofs**,** it is easily seen that the sets in the Table 4 , Table 5 and Table 6 are determined till 5000 and so on…

Now, choosing some pairs from the mentioned above we can try to determine their extendibility to Diophantine D(400) using solutions of the Pell/Pell like equations as follows:

Theorem 5. The set {2} is extended to Diophantine *D(400)* triples with positive integers given by the following Table 7.

Table 7. Extendibility of $\{2\}$ to Diophantine *D(400)* triples up to 3000.

{2, 42, 88}	{2, 192, 952}	{2, 378, 448}	{2, 952, 1050}	{2, 1722, 1848}
{2, 42, 192}	{2, 192, 1848}	{2, 448, 522}	{2, 1050, 1152}	{2, 1848, 1978}
{2, 42, 600}	{2, 250, 312}	{2, 522, 600}	{2, 1152, 1258}	{2, 1978, 2112}
{2, 42, 1600}	{2, 250, 1848}	{2, 600, 682}	{2, 1258, 1368}	{2, 2112, 2250}
{2, 88, 138}	{2, 250, 2688}	{2, 600, 1600}	{2, 1368, 1482}	{2, 2250, 2392}
{2, 138, 192}	{2, 312, 378}	{2, 682, 768}	{2, 1482, 1600}	{2, 2392, 2538}
{2, 192, 250}	{2, 312, 2112}	{2, 768, 858}	{2, 1482, 2688}	{2, 2538, 2688}
{2, 192, 600}	{2, 312, 2688}	{2, 858, 952}	{2, 1600, 1722}	{2, 2688, 2842}

Proof. Let Q be the third positive integer of the set $\{2, 42\}$. So, $\{2, 42, Q\}$ is known as Diophantine *D(400)* triple. Then, the following equations have solutions (m, n are integers) from the definition of Diophantine *D(400)* triple:

$2Q + 400 = m^2$ and $42Q + 400 = n^2$

Dropping Q from the equations, we get a Pell equation as follows:

$21m^2 - n^2 = 8000.$

If we search fundamental solutions of the Pell equation, (Discriminant =84), we obtain the following 12 (twelve) different fundamental solution families.

Least positive solution of $u^2 - 84v^2 = 4$: $(\varphi_1,\psi_1)=(110,12)$

Fundamental solution [0]: (-28, 92) , m = (-28u + 184v)/2, n = (92u - 1176v)/2

Fundamental solution [1]: (28, 92), m = (28u + 184v)/2, n = (92u + 1176v)/2

Fundamental solution [2]: (-92, 412) , m = (-92u + 824v)/2, n = (412u - 3864v)/2

Fundamental solution [3]: (92, 412) , m = (92u + 824v)/2, n = (412u + 3864v)/2

Fundamental solution [4]: (-20, 20) , m = (-20u + 40v)/2 , n = (20u - 840v)/2

Fundamental solution [5]: (20, 20) , m = (20u + 40v)/2, n = (20u + 840v)/2

Fundamental solution [6]: (-60, 260) , m = (-60u + 520v)/2 , n = (260u - 2520v)/2

Fundamental solution [7]: (60, 260) , m = (60u + 520v)/2 , n = (260u + 2520v)/2

Fundamental solution [8]: (-24, 64) , m = (-24u + 128v)/2 , n = (64u - 1008v)/2

Fundamental solution [9]: (24, 64) , m = (24u + 128v)/2 ,n = (64u + 1008v)/2

Fundamental solution [10]: (-40, 160) , m = (-40u + 320v)/2, n = (160u - 1680v)/2

Fundamental solution [11]: (40, 160) , m = (40u + 320v)/2, n = (160u + 1680v)/2

Using them, we obtain Q=88 is the smallest positive integer that makes the left-hand side of the equation a perfect square and others are 192, 600, 1600, so on..

In the same way and similar calculation, other Diophantine D(400) triples are obtained.

Additionally, one may prove whether or not there are regular triple using the regularity condition from Preliminaries section.

Theorem 6. The set $\{3\}$ is extended to Diophantine *D(400)* triples with positive integers given by the following Table 8.

Table 8. Extendibility of $\{3\}$ to Diophantine *D(400)* triples till 3000.

{3, 28, 75}	{3, 75, 275}	{3, 187, 252}	{3, 275, 483}	{3, 427, 1547}	{3, 768, 875}	{3, 1275, 1408}	{3, 2000, 2163}
{3, 28, 275}	{3, 75, 700}	{3, 187, 875}	{3, 275, 768}	{3, 483, 572}	{3, 803, 912}	{3, 1363, 1500}	{3, 2108, 2275}
{3, 28, 483}	{3, 92, 147}	{3, 187, 1500}	{3, 275, 2688}	{3, 483, 912}	{3, 875, 988}	{3, 1408, 1547}	{3, 2163, 2332}
{3, 28, 768}	{3, 128, 187}	{3, 208, 275}	{3, 323, 400}	{3, 483, 1275}	{3, 912, 1027}	{3, 1500, 1643}	{3, 2275, 2448}
{3, 28, 1275}	{3, 128, 348}	{3, 208, 912}	{3, 323, 2688}	{3, 512, 603}	{3, 988, 1107}	{3, 1547, 1692}	{3, 2332, 2507}
{3, 43, 92}	{3, 128, 700}	{3, 208, 2688}	{3, 348, 427}	{3, 572, 667}	{3, 988, 2163}	{3, 1643, 1792}	{3, 2448, 2627}
{3, 43, 252}	{3, 128, 2275}	{3, 252, 323}	{3, 348, 1547}	{3, 603, 700}	{3, 1027, 1148}	{3, 1692, 1843}	{3, 2507, 2688}
{3, 43, 875}	{3, 147, 187}	{3, 252, 875}	{3, 348, 2275}	{3, 603, 2332}	{3, 1107, 1232}	{3, 1792, 1947}	{3, 2627, 2812}
{3, 43, 1275}	{3, 147, 208}	{3, 252, 1408}	{3, 400, 483}	{3, 667, 768}	{3, 1148, 1275}	{3, 1843, 2000}	{3, 2688, 2875}
{3, 75, 128}	{3, 147, 1500}	{3, 275, 348}	{3, 427, 512}	{3, 700, 803}	{3, 1232, 1363}	{3, 1947, 2108}	

Proof. Let R be the third positive integer of the set {3,75}. So, {3,75, R } is known as Diophantine *D(400)* triple. Then, the following equations have solutions (U, V are integers) from the definition of Diophantine *D(400)* triple:

$3R + 400 = U^2$ and $75R + 400 = V^2$

Dropping R from the equations, we get a Pell equation as follows:

$25U^2 - V^2 = 9600.$

If we search fundamental solutions of the Pell equation, (Discriminant =100 perfect square), then we obtain 24 (twenty four) different solutions as follows:

By solving (5U + V)(5U - V) = 9600, it gives finitely many solutions due to factorising the right side:

solution [0]: (97,-475) , solution[1]: (35,-145), solution[2]: (50,-230)

solution[3]: (22,-50) , solution[4]: (28,-100) , solution[5]: (20,20)

solution[6]: (20,-20) , solution[7]: (28,100), solution[8]: (22,50)

solution[9]: (50,230), solution[10]: (35,145), solution[11]: (97,475)

solution[12]: (-97,475), solution[13]: (-35,145). solution[14]: (-50,230)

solution[15]: (-22,50), solution[16]: (-28,100), solution[17]: (-20,-20)

solution[18]: (-20,20) , solution[19]: (-28,-100) , solution[20]: (-22,-50)

solution[21]: (-50,-230) , solution[22]: (-35,-145) , solution[23]: (-97,-475)

Using them, we obtain R=128 is the smallest positive integer that makes the left-hand side of the equation a perfect square and others are 275, 700, so on. So, $\{3, 75, 128\}$ is Diophantine D(400) triple.

Similarly, other Diophantine D(400) triples are obtained from $\{3\}$ to Diophantine *D(400)*. Besides, we should prove whether or not $\{3, 75, 128\}$ is Regular Diophantine *D(400)* triple?

Using regularity condition given in preliminaries section for Diophantine *D(s)* triples, it is seen that $((128-75-3)^2 = 4.(3.75+400))$ holds and it is regular Diophantine *D(400)* triples. Thus, the proof is completed.

Theorem 7. (a) The set $\{4\}$ is extended to Diophantine *D(400)* triples with positive integers given by the following Table 9.

Table 9. Extendibility of $\{4\}$ to Diophantine *D(400)* triples till value 3000.

{4, 21, 69}	{4, 69, 384}	{4, 125, 1581}	{4, 341, 429}	{4, 629, 741}	{4, 1125, 2816}	{4, 1925, 2109}
{4, 21, 125}	{4, 69, 741}	{4, 125, 2925}	{4, 384, 476}	{4, 684, 800}	{4, 1196, 1344}	{4, 2016, 2204}
{4, 21, 384}	{4, 69, 1925}	{4, 156, 224}	{4, 384, 924}	{4, 684, 2816}	{4, 1269, 1421}	{4, 2109, 2301}
{4, 21, 429}	{4, 96, 156}	{4, 189, 261}	{4, 384, 1925}	{4, 741, 861}	{4, 1344, 1500}	{4, 2204, 2400}
{4, 21, 1125}	{4, 96, 224}	{4, 189, 1581}	{4, 384, 2925}	{4, 800, 924}	{4, 1421, 1581}	{4, 2301, 2501}
{4, 21, 1664}	{4, 96, 576}	{4, 224, 300}	{4, 429, 525}	{4, 861, 989}	{4, 1500, 1664}	{4, 2400, 2604}
{4, 21, 2816}	{4, 96, 1500}	{4, 224, 1500}	{4, 429, 1125}	{4, 924, 1056}	{4, 1581, 1749}	{4, 2501, 2709}
{4, 44, 96}	{4, 125, 189}	{4, 224, 2604}	{4, 476, 576}	{4, 989, 1125}	{4, 1664, 1836}	{4, 2604, 2816}
{4, 44, 1344}	{4, 125, 384}	{4, 261, 341}	{4, 525, 629}	{4, 1056, 1196}	{4, 1749, 1925}	{4, 2709, 2925}
{4, 69, 125}	{4, 125, 741}	{4, 300, 384}	{4, 576, 684}	{4, 1125, 1269}	{4, 1836, 2016}	

(b) The set $\{5\}$ is extended to Diophantine *D(400)* triples with positive integers given by the following Table 10.

Table 10. Extendibility of $\{5\}$ to Diophantine *D(400)* triples up to value 3000.

{5, 45, 100}	{5, 165, 1365}	{5, 640, 765}	{5, 1725, 1920}
{5, 45, 525}	{5, 165, 2800}	{5, 765, 900}	{5, 1920, 2125}
{5, 100, 165}	{5, 240, 325}	{5, 900, 1045}	{5, 2125, 2340}
{5, 100, 525}	{5, 240, 2800}	{5, 1045, 1200}	{5, 2340, 2565}
{5, 100, 1365}	{5, 325, 420}	{5, 1200, 1365}	{5, 2565, 2800}
{5, 165, 240}	{5, 420, 525}	{5, 1365, 1540}	
{5, 165, 525}	{5, 525, 640}	{5, 1540, 1725}	

(c) The set $\{7\}$ is extended to Diophantine *D(400)* triples with positive integers given by the following Table 11.

Table 11. Extendibility of $\{7\}$ to Diophantine *D(400)* triples up to 3000.

{7, 12, 63}	{7, 63, 2508}	{7, 207, 663}	{7, 492, 623}	{7, 975, 1152}	{7, 1767, 2000}
{7, 12, 128}	{7, 108, 183}	{7, 272, 375}	{7, 528, 663}	{7, 975, 2432}	{7, 1932, 2175}
{7, 12, 375}	{7, 128, 207}	{7, 272, 1287}	{7, 623, 663}	{7, 1100, 1287}	{7, 2000, 2247}
{7, 12, 975}	{7, 128, 375}	{7, 300, 407}	{7, 623, 768}	{7, 1152, 1343}	{7, 2175, 2432}
{7, 12, 2432}	{7, 128, 663}	{7, 375, 492}	{7, 663, 8120	{7, 1152, 1932}	{7, 2247, 2508}
{7, 47, 108}	{7, 128, 1767}	{7, 375, 975}	{7, 768, 927}	{7, 1287, 14880	{7, 2432, 2703}
{7, 47, 975}	{7, 183, 272}	{7, 375, 1287}	{7, 768, 2508}	{7, 1343, 1548}	{7, 2508, 2783}
{7, 47, 1932}	{7, 183, 812}	{7, 375, 2508}	{7, 812, 975}	{7, 1488, 1703}	{7, 2703, 2988}
{7, 63, 128}	{7, 183, 1767}	{7, 407, 528}	{7, 812, 1767}	{7, 1548, 1767}	
{7, 63, 375}	{7, 207, 300}	{7, 407, 1548}	{7, 927, 1100}	{7, 1703, 1932}	

Proof. In the same manner of the proof of the previous last two theorems, it is easily seen their proof.

3. Discussion and Conclusion

Diophantine set theory, which has a very important place in number theory and its applications in other fields (physics, chemistry, engineering, cryptology, knowledge discovery, data science and etc.), is a special type of Diophantine equation. Today, it is an important subject that is still actively studied and that can form a chain link thanks to its connections to different fields.

This study represents how some numerical results can be obtained on Diophantine clusters with D(400) properties, where different methods are used together. In the next work, such sets will be considered again and searched their extendibility from triple to quadruple as Diophantine D(400) set. Using software, it will be fast and efficent to calculate numerical results easily, but we should have to get powerful computer. So, we tried to give mathematical steps to calculate them in this study. Thus, we supported to the literature as mathematically in the related field.

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